流体力学特論

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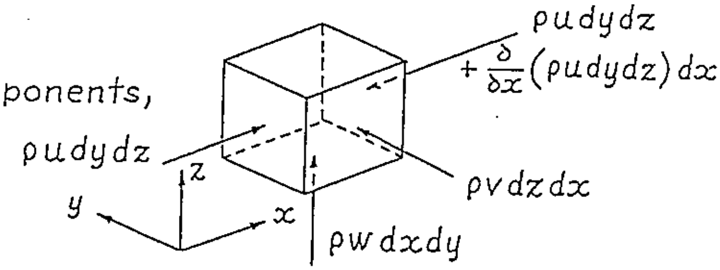
CONTENTS

Navier-Stokes equations

- Mass conservation.

  u, v, w : velocity components,

  ρ : density.



$$\frac{\partial}{\partial x}(\rho u\,dy\,dz)\,dx + \frac{\partial}{\partial y}(\rho v\,dz\,dx)\,dy + \frac{\partial}{\partial z}(\rho w\,dx\,dy)\,dz = 0.$$

for incompressible flows (ρ = const):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{(continuity equation)}.$$

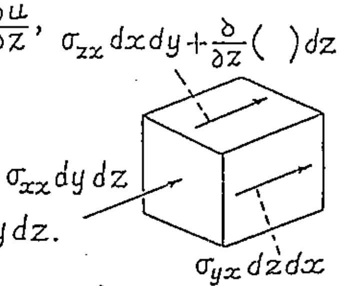
- Momentum conservation.

  mass $\times$ acceleration = net force due to stresses;

  mass = $\rho\,dx\,dy\,dz$, t : time,

  accel. component $\dfrac{du}{dt} = \lim\limits_{\Delta t \to 0} \dfrac{u(t+\Delta t, x+u\Delta t, y+v\Delta t, z+w\Delta t) - u(t,x,y,z)}{\Delta t}$

$$= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$



net force component due to surface stresses:

$$\frac{\partial}{\partial x}(\sigma_{xx}\,dy\,dz)\,dx + \frac{\partial}{\partial y}(\sigma_{yx}\,dz\,dx)\,dy$$
$$+ \frac{\partial}{\partial z}(\sigma_{zx}\,dx\,dy)\,dz = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right) dx\,dy\,dz.$$

difference in velocity component $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy + \dfrac{\partial u}{\partial z}dz$

$$= \qquad -\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz$$
$$+ \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dz$$
$$= -\frac{1}{2}\omega_z\,dy + \frac{1}{2}\omega_y\,dz + \frac{1}{2}s_{xx}\,dx + \frac{1}{2}s_{yx}\,dy + \frac{1}{2}s_{zx}\,dz$$

= antisymmetric rotation + symmetric deformation.

$\frac{1}{2}\omega_y$: rate of rotation about axis parallel to y-axis,

$\frac{1}{2}s_{xx}$: rate of extension of a line element along x-axis,

$\frac{1}{2}s_{yx}$: rate of change of angle between line elements along y- and x-axis.

relation between stress and rate of deformation:

$$\sigma_{xx} = -p + \mu s_{xx}, \quad \sigma_{yx} = \mu s_{yx}, \quad \sigma_{zx} = \mu s_{zx}; \qquad p: \text{pressure}, \ \mu: \text{viscosity}.$$

$$\begin{Bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{Bmatrix} = \begin{Bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{Bmatrix} + \mu \begin{Bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{Bmatrix}.$$

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} &= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{yx}}{\partial y} + \frac{\partial s_{zx}}{\partial z}\right) \\ &= -\frac{\partial p}{\partial x} + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ &= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ &= -\frac{\partial p}{\partial x} + \rho\nu\nabla^2 u; \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \end{aligned}$$

$$\nu = \frac{\mu}{\rho}: \text{ kinematic viscosity}.$$

- Navier-Stokes equations with external force f_x, f_y, f_z per unit mass:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= f_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= f_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w. \end{aligned}$$

Along with the continuity equation, these equations make it possible to determine u, v, w and p as functions of t, x, y and z under certain initial and boundary conditions.

- Vector form of equations: $\vec{u}(u, v, w)$, $\nabla\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$;

$$\nabla\vec{u} = 0, \quad \frac{\partial \vec{u}}{\partial t} + (\vec{u}\nabla)\vec{u} = \vec{f} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u}.$$

- Tensor form of equations: $(\alpha, \beta = 1, 2, 3)$

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0, \quad \frac{\partial u_\alpha}{\partial t} + u_\beta\frac{\partial u_\alpha}{\partial x_\beta} = f_\alpha - \frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \nu\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta}.$$

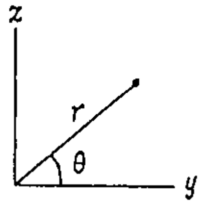
$$du_\alpha = \frac{\partial u_\alpha}{\partial x_\beta}dx_\beta = \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\,\omega_\beta\,dx_\gamma + \frac{1}{2}s_{\alpha\beta}\,dx_\beta,$$

$$\omega_\alpha = \varepsilon_{\alpha\beta\gamma}\frac{\partial u_\gamma}{\partial x_\beta}, \quad s_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha},$$

$\sigma_{\alpha\beta} = -p\,\delta_{\alpha\beta} + \mu s_{\alpha\beta}$; $\delta_{\alpha\beta}$ Kronecker δ, $\varepsilon_{\alpha\beta\gamma}$ permutation tensor.

Unidirectional flows

- Poiseuille flow: $$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$



$v = w = 0;\ \frac{\partial u}{\partial x} = 0:$ $$0 = -\frac{\partial p}{\partial y} = -\frac{\partial p}{\partial z}.$$

$$-\frac{\partial p}{\partial x} = G: \quad \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{G}{\mu}.$$

$$\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right);$$

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = -\frac{G}{\mu}r, \quad u = \frac{G}{4\mu}(a^2 - r^2), \quad Q = \int_0^a u\cdot 2\pi r\,dr = \frac{\pi a^4 G}{8\mu}.$$

Turbulent for $\frac{Q}{\pi a^2}\frac{a}{\nu} > 10^3$.

- Two-dimensional unsteady flow $\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$, $u = u(y,t)$.

a) Oscillating plate, $u(0,t) = U\cos\omega t$.

$$u = \Re\{e^{i\omega t}F(y)\}: \quad i\omega F = \nu F''; \quad (') = \frac{d}{dy}, \quad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$F = Ae^{\sqrt{i\frac{\omega}{\nu}}y} + Be^{-\sqrt{i\frac{\omega}{\nu}}y} = Ae^{(1+i)\sqrt{\frac{\omega}{2\nu}}y} + Be^{-(1+i)\sqrt{\frac{\omega}{2\nu}}y} = Be^{-(1+i)ky} \quad \left(k = \sqrt{\frac{\omega}{2\nu}}\right)$$

$$u = \Re Be^{-ky}e^{i(\omega t - ky)} = Be^{-ky}\cos(\omega t - ky) = Ue^{-ky}\cos(\omega t - ky)$$

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = -\mu U k e^{-ky}\{\cos(\omega t - ky) - \sin(\omega t - ky)\}_{y=0} = -\sqrt{2}k\mu U\cos\left(\omega t + \frac{\pi}{4}\right).$$

b) Impulsively started plate, $u(0,t) = U$ for $t > 0$, $u(y,0) = 0$ for $y > 0$.

$$\sigma = \frac{y}{2\sqrt{\nu t}}: \quad \frac{\partial u}{\partial t} = -\frac{\sigma}{2t}\frac{du}{d\sigma}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{(2\sqrt{\nu t})^2}\frac{d^2 u}{d\sigma^2}; \quad \frac{d^2 u}{d\sigma^2} = -2\sigma\frac{du}{d\sigma},$$

$$\frac{du}{d\sigma} = -Ce^{-\sigma^2}, \quad u = C\int_\sigma^\infty e^{-\tau^2}d\tau = \frac{2U}{\sqrt{\pi}}\int_\sigma^\infty e^{-\tau^2}d\tau = U\,\mathrm{erfc}\,\sigma.$$

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = -\frac{\mu U}{\sqrt{\pi\nu t}}. \qquad \sigma = 1.82,\ \mathrm{erfc}\,\sigma = 0.01: \quad y_{0.01} = 3.64\sqrt{\nu t}.$$

c) Numerical solution

$$\frac{u(y,t+k) - u(y,t)}{k} = \nu\frac{u(y+h,t) - 2u(y,t) + u(y-h,t)}{h^2}.$$

$$\xi = \frac{\nu k}{h^2}: \quad u(y,t+k) = \xi u(y+h,t) + (1-2\xi)u(y,t) + \xi u(y-h,t);$$

$\xi < \frac{1}{2}$ for stability, $\xi = \frac{1}{6}$ for minimizing truncation error;

$$\xi = \frac{1}{6}: \quad u(y,t+k) = \boxed{\begin{array}{c|c|c} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{array}}\ u(y,t).$$

Reynolds number

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0, \quad \frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} = -\frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \nu \frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta}; \quad \left.\begin{matrix} u_\alpha = 0 \text{ on } S \\ u_\alpha = U_0 \text{ at } \infty \end{matrix}\right\}.$$

$$x_\alpha = L x_\alpha^*, \quad u_\alpha = U_0 u_\alpha^*, \quad t = \frac{L}{U_0} t^*, \quad p = \rho U_0^2 p^* :$$

$$\frac{\partial u_\alpha^*}{\partial x_\alpha^*} = 0, \quad \frac{\partial u_\alpha^*}{\partial t^*} + u_\beta^* \frac{\partial u_\alpha^*}{\partial x_\beta^*} = -\frac{\partial p^*}{\partial x_\alpha^*} + \frac{1}{R}\frac{\partial^2 u_\alpha^*}{\partial x_\beta^* \partial x_\beta^*}; \quad \left.\begin{matrix} u_\alpha^* = 0 \text{ on } S \\ u_\alpha^* = 1 \text{ at } \infty \end{matrix}\right\}.$$

$$R = \frac{U_0 L}{\nu} : \text{ Reynolds number}$$

$R \ll 1$: viscosity dominated; Stokes approximation.

$R \gg 1$: inertia dominated; boundary layer approximation.

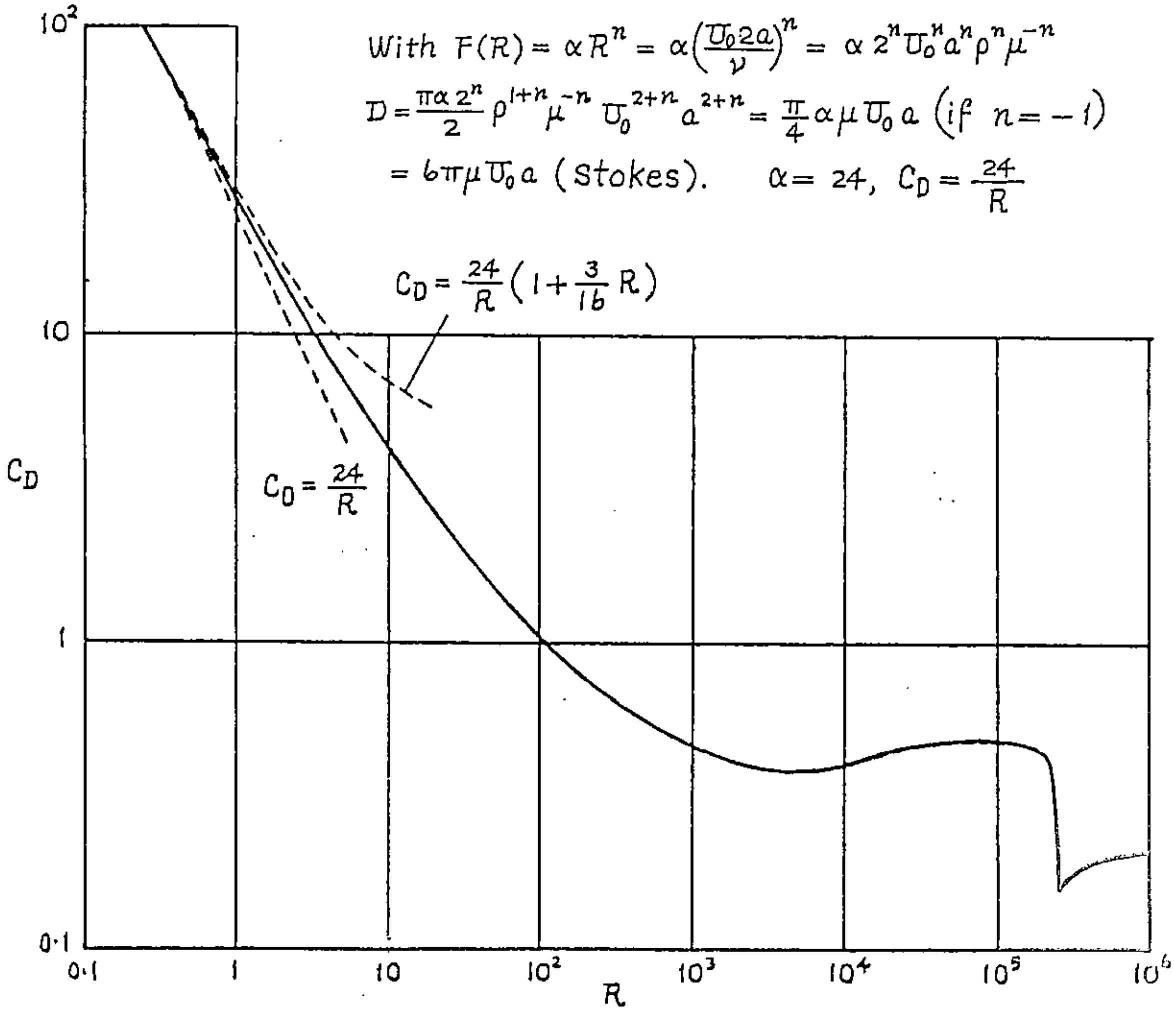
Flow past a sphere

Drag $D = \frac{1}{2}\rho U_0^2 \pi a^2 C_D, \quad C_D = F(R); \quad R = \frac{U_0 \cdot 2a}{\nu}.$

With $F(R) = \alpha R^n = \alpha\left(\frac{U_0 2a}{\nu}\right)^n = \alpha\, 2^n U_0^n a^n \rho^n \mu^{-n}$

$D = \frac{\pi\alpha 2^n}{2}\rho^{1+n}\mu^{-n} U_0^{2+n} a^{2+n} = \frac{\pi}{4}\alpha\mu U_0 a$ (if $n = -1$)

$= 6\pi\mu U_0 a$ (Stokes). $\quad \alpha = 24, \; C_D = \frac{24}{R}$



Vorticity equations

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \omega_\alpha = \varepsilon_{\alpha\beta\gamma}\frac{\partial u_\gamma}{\partial x_\beta},$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \varepsilon_{\alpha\beta\gamma} : \text{ permutation tensor.}$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

$$\left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\left\{\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u,\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v,\\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w.
\end{aligned}\right.
\quad
\begin{array}{|ccc}
 & +\frac{\partial}{\partial z} & -\frac{\partial}{\partial y}\\
-\frac{\partial}{\partial z} & & +\frac{\partial}{\partial x}\\
+\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & \\ \hline
\end{array}$$

$$\left\{\begin{aligned}
\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + w\frac{\partial \xi}{\partial z} &= \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} + \nu\nabla^2\xi,\\
\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} + w\frac{\partial \eta}{\partial z} &= \xi\frac{\partial v}{\partial x} + \eta\frac{\partial v}{\partial y} + \zeta\frac{\partial v}{\partial z} + \nu\nabla^2\eta,\\
\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} &= \xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} + \zeta\frac{\partial w}{\partial z} + \nu\nabla^2\zeta.
\end{aligned}\right.$$

$$\frac{\partial \omega_\alpha}{\partial t} + u_\beta\frac{\partial \omega_\alpha}{\partial x_\beta} = \omega_\beta\frac{\partial u_\alpha}{\partial x_\beta} + \nu\frac{\partial^2\omega_\alpha}{\partial x_\beta\partial x_\beta}.$$

- Two-dimensional flow

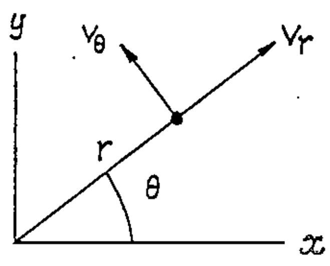
$w = 0,\ \frac{\partial}{\partial z} = 0 :\quad \xi = \eta = 0,\quad \zeta$: single component of vorticity.

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 :\quad u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x}\ ;\quad \psi$: stream function.

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \nu\nabla^2\zeta, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2\psi, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

If $\zeta = f(\psi)$, $\ u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \frac{\partial \psi}{\partial y}\frac{df}{d\psi}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{df}{d\psi}\frac{\partial \psi}{\partial y} = 0, \quad \frac{\partial \zeta}{\partial t} = \nu\nabla^2\zeta.$

- Polar coordinates $(r, \theta\ ;\ v_r, v_\theta)$:



$$\left.\begin{aligned} x &= r\cos\theta,\\ y &= r\sin\theta.\end{aligned}\right\} \quad \begin{aligned} &\frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta,\\ &\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.\end{aligned}$$

$$\left.\begin{aligned} u &= v_r\cos\theta - v_\theta\sin\theta,\\ v &= v_r\sin\theta + v_\theta\cos\theta.\end{aligned}\right\}$$

$$\begin{aligned}
\zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} - \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} - \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}\\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)(v_r\sin\theta + v_\theta\cos\theta) - \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)(v_r\cos\theta - v_\theta\sin\theta)\\
&= \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}
\end{aligned}$$

$$\boxed{\begin{aligned} &\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_\theta}{\partial \theta} = 0; \quad \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}; \\ &v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right), \\ &v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2}\right). \end{aligned}}$$

- Circular flow (steady):

$$v_r = 0,\ v_\theta = q(r,\theta) = q(r),\ \frac{\partial}{\partial\theta} = 0,\ \nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr};$$

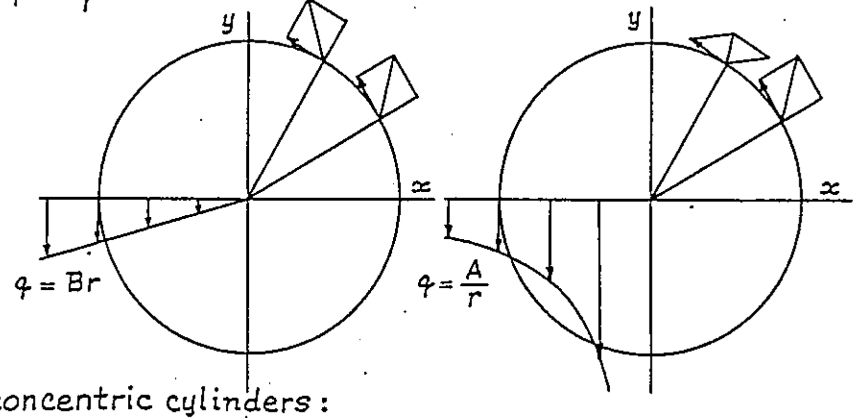
$$\zeta = \frac{1}{r}\frac{d}{dr}(rq) = \frac{dq}{dr} + \frac{q}{r} = q' + \frac{q}{r},\ (') = \frac{d}{dr}.$$

$$\frac{\partial\zeta}{\partial t} = 0 = \nu\nabla^2\zeta: \quad \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left(q' + \frac{q}{r}\right)\right\} = q''' + 2\frac{q''}{r} - \frac{q'}{r^2} + \frac{q}{r^3} = 0,$$

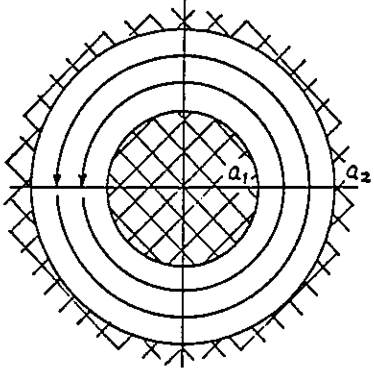
$$q = r^n: \quad (n+1)(n-1)^2 = 0, \quad q = \frac{A}{r} + Br + Cr\log r.$$

$$\nabla^2 v_\theta - \frac{v_\theta}{r^2} = q'' + \frac{q'}{r} - \frac{q}{r^2} = 2\frac{C}{r} = 0: \quad C = 0.$$

$$q = \frac{A}{r} + Br, \quad \zeta = 2B = \text{constant}.$$



concentric cylinders:



$$\left.\begin{aligned} r = a_1: \quad q = \frac{A}{a_1} + Ba_1 = \omega_1 a_1, \\ r = a_2: \quad q = \frac{A}{a_2} + Ba_2 = \omega_2 a_2. \end{aligned}\right\}$$

$$A = -\frac{(\omega_2 - \omega_1)a_1^2 a_2^2}{a_2^2 - a_1^2},$$

$$B = \frac{\omega_2 a_2^2 - \omega_1 a_1^2}{a_2^2 - a_1^2}.$$

$$\tau = -\rho\nu\left\{r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right\} = -\rho\nu\left(q' - \frac{q}{r}\right) = 2\rho\nu\frac{A}{r^2};$$

$$M_1 = 2\rho\nu\frac{A}{a_1^2}a_1\cdot 2\pi a_1 = 4\pi\rho\nu A, \quad M_2 = -2\rho\nu\frac{A}{a_2^2}a_2\cdot 2\pi a_2 = -M_1.$$

Rankine vortex:

$0 \leqslant r \leqslant a: \quad q = Br = \frac{1}{2}\zeta_0 r;$

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{q^2}{r},$$

$$\frac{p}{\rho} = \frac{p_0}{\rho} + \int_0^r \frac{q^2}{r} = \frac{p_0}{\rho} + \frac{1}{2}B^2 r^2.$$

$a \leqslant r < \infty: \quad q = \frac{A}{r};$

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \frac{p_\infty}{\rho} \text{ (Bernoulli)},$$

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} - \frac{1}{2}\frac{A^2}{r^2}.$$

$$\left.\begin{aligned} r = a: \quad & Ba = \frac{A}{a}, \quad \frac{p_0}{\rho} + \frac{1}{2}B^2a^2 = \frac{p_\infty}{\rho} - \frac{1}{2}\frac{A^2}{a^2}; \\ & \frac{p_0}{\rho} = \frac{p_\infty}{\rho} - B^2a^2 = \frac{p_\infty}{\rho} - \frac{1}{4}\zeta_0^2 a^2. \end{aligned}\right\}$$

$$\kappa = \oint q\,ds = \frac{A}{r}\,2\pi r = 2\pi A = 2\pi Ba^2,$$

$$\iint \zeta\, dS = \zeta_0 \pi a^2 = 2\pi Ba^2 = \kappa.$$

- *Unsteady circular flow:*

Suppose we have initially an isolated vortex of strength κ concentrated at $r=0$. The solution of $\frac{\partial\zeta}{\partial t} = \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\zeta}{\partial r}\right) = \nu\left(\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r}\right)$ is

$$\zeta = \frac{\kappa}{4\pi\nu t}\, e^{-\frac{r^2}{4\nu t}},$$

as easily verified by differentiation. It gives for the circulation in a circle of radius r the value

$$\int_0^r \zeta \cdot 2\pi r\,dr = \kappa\left(1 - e^{-\frac{r^2}{4\nu t}}\right),$$

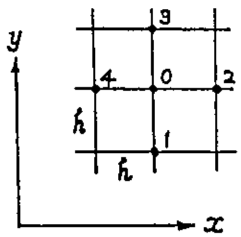
the limiting value of which for $t \to 0$ is κ. The velocity is

$$q = \frac{\kappa}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right).$$

Numerical solution of two-dimensional flows

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

$$\left.\begin{aligned} \frac{\partial \zeta}{\partial t} &= \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x} + \nu \nabla^2 \zeta, \\ \zeta &= -\nabla^2 \psi, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \end{aligned}\right\}$$



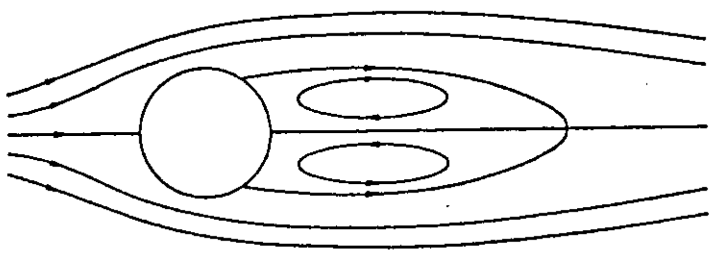
$$\frac{\partial \zeta}{\partial x} = \frac{\zeta_2 - \zeta_4}{2h}, \quad \frac{\partial^2 \zeta}{\partial x^2} = \frac{\zeta_2 - 2\zeta_0 + \zeta_4}{h^2},$$

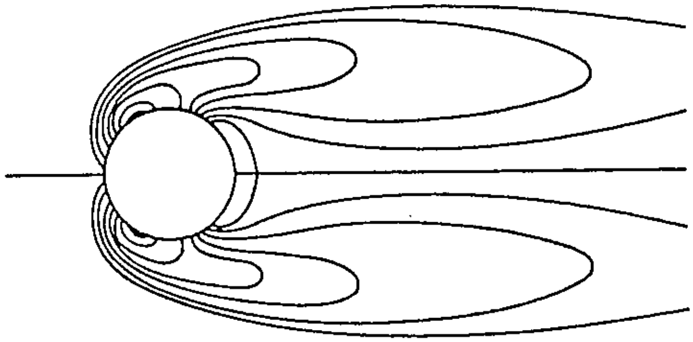
$$\nabla^2 \zeta = \frac{\zeta_2 - 2\zeta_0 + \zeta_4}{h^2} + \frac{\zeta_3 - 2\zeta_0 + \zeta_1}{h^2} = \frac{\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 - 4\zeta_0}{h^2};$$

$$\frac{\partial \zeta}{\partial t} = \frac{\zeta_0(t+k) - \zeta_0(t)}{k} = \frac{\zeta_0^1 - \zeta_0^0}{k}.$$

$$\left.\begin{aligned} \frac{\zeta_0^1 - \zeta_0}{k} &= \frac{(\psi_2 - \psi_4)(\zeta_3 - \zeta_1) - (\psi_3 - \psi_1)(\zeta_2 - \zeta_4)}{4h^2} + \nu \frac{\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 - 4\zeta_0}{h^2}, \\ \zeta_0 &= -\frac{\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0}{h^2}. \end{aligned}\right\}$$

$$\frac{\partial}{\partial t} = 0: \quad \left.\begin{aligned} \zeta_0 &= \frac{\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4}{4} + \frac{(\psi_2 - \psi_4)(\zeta_3 - \zeta_1) - (\psi_3 - \psi_1)(\zeta_2 - \zeta_4)}{16\nu}, \\ \psi_0 &= \frac{\psi_1 + \psi_2 + \psi_3 + \psi_4}{4} + \frac{h^2}{4}\zeta_0. \end{aligned}\right\}$$



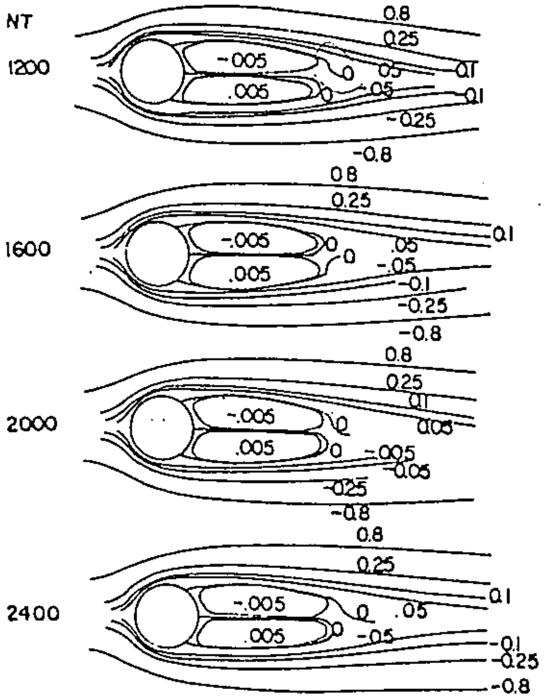


$$\frac{2aU}{\nu} = 40$$

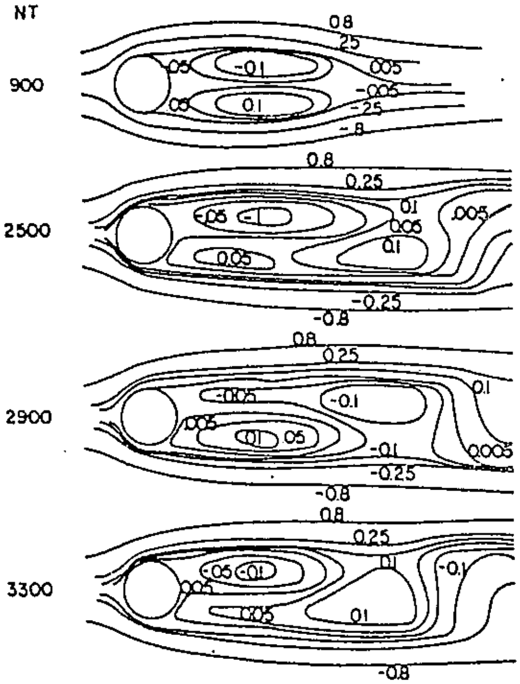
Takami & Keller 1969

SEPARATED FLOW SOLUTIONS AROUND A CIRCULAR CYLINDER

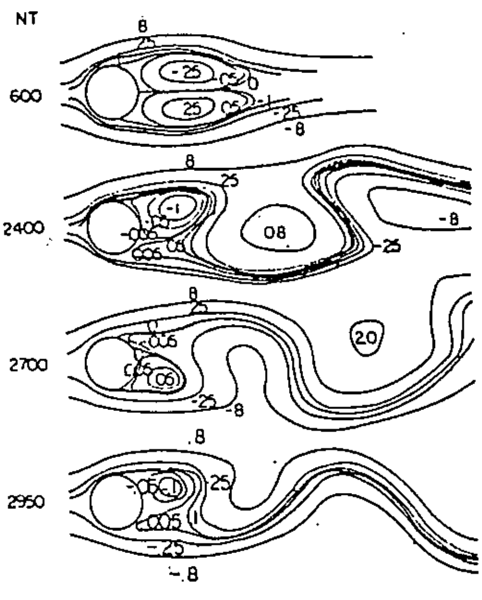
Lin, Pepper & Lee, 1976. Reynolds number: $2aU/\nu$,
NT: number of time steps, time step $\Delta(Ut/a)$: 0.02.



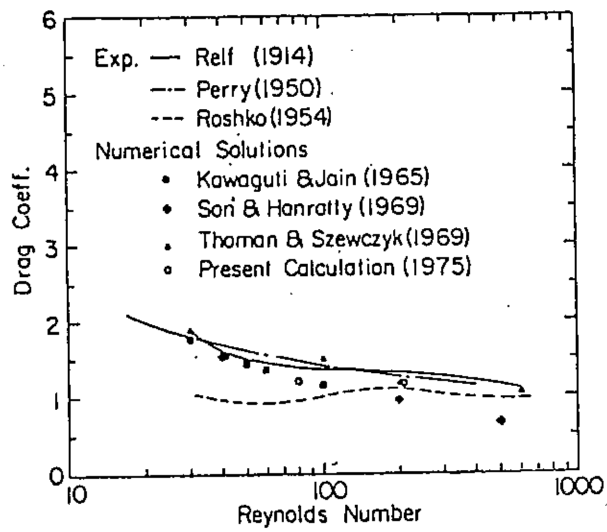
Reynolds number 40



Reynolds number 80

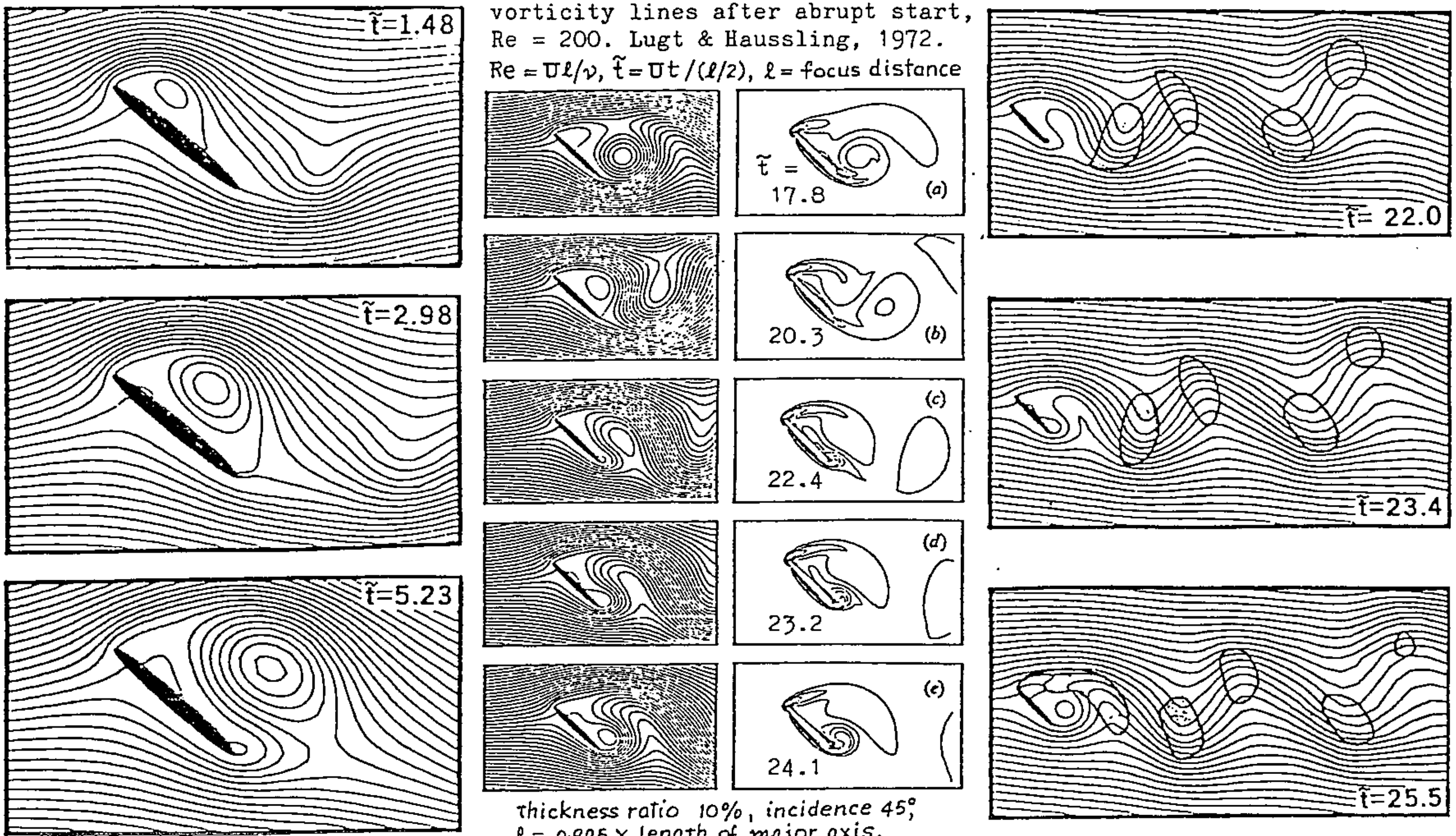


Reynolds number 200

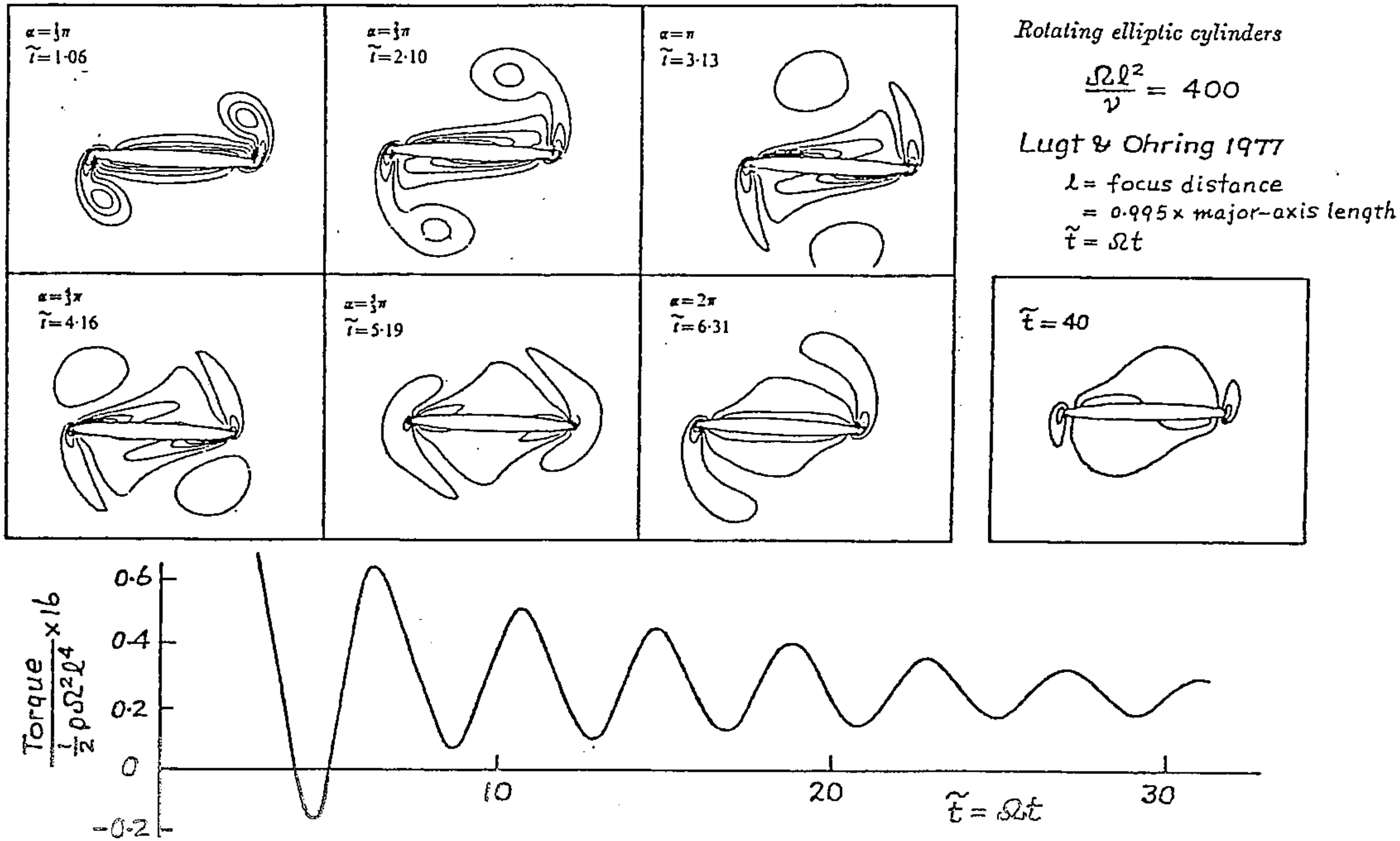


Comparison with Experimental Data

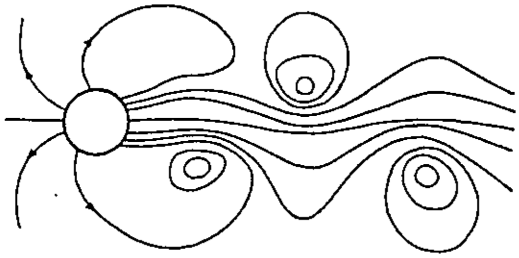
Sequence of streamlines and equi-vorticity lines after abrupt start, Re = 200. Lugt & Haussling, 1972.
$Re = U\ell/\nu$, $\tilde{t} = Ut/(\ell/2)$, ℓ = focus distance

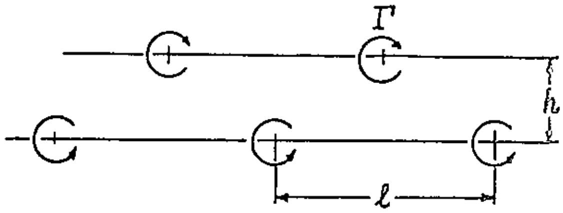


thickness ratio 10%, incidence 45°,
ℓ = 0·995 x length of major axis.


Rotating elliptic cylinders
$\frac{\Omega \ell^2}{\nu} = 400$
Lugt & Ohring 1977
ℓ = focus distance
= 0·995 × major–axis length
$\tilde{t} = \Omega t$
$\alpha = \frac{1}{3}\pi$
$\tilde{t} = 1{\cdot}06$
$\alpha = \frac{2}{3}\pi$
$\tilde{t} = 2{\cdot}10$
$\alpha = \pi$
$\tilde{t} = 3{\cdot}13$
$\alpha = \frac{4}{3}\pi$
$\tilde{t} = 4{\cdot}16$
$\alpha = \frac{5}{3}\pi$
$\tilde{t} = 5{\cdot}19$
$\alpha = 2\pi$
$\tilde{t} = 6{\cdot}31$
$\tilde{t} = 40$
$\frac{\text{Torque}}{\frac{1}{2}\rho\Omega^2\ell^4} \times 16$
0·6
0·4
0·2
0
−0·2
10
20
30
$\tilde{t} = \Omega t$

von Kármán vortex street





condition of stability $\quad \sinh \frac{\pi h}{\ell} = 1$

velocity of cylinder $\quad U$

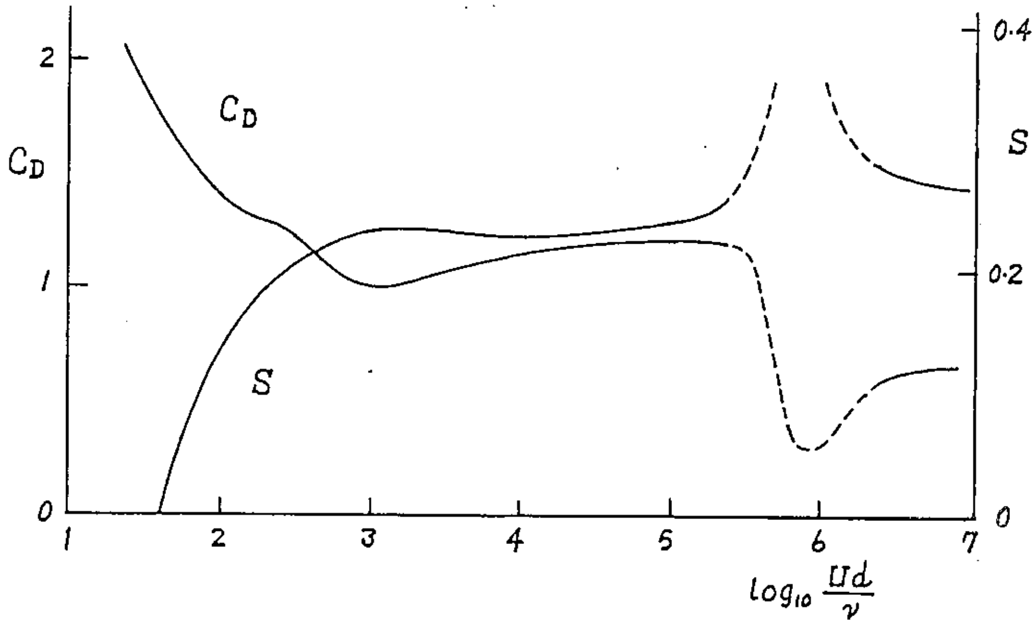
velocity of vortex system $\quad u = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \frac{\Gamma h}{h^2 + (2m+1)^2 \frac{\ell^2}{4}}$

$$= \frac{\Gamma}{2\ell} \tanh \frac{\pi h}{\ell} = \frac{\Gamma}{2\ell} \frac{1}{\sqrt{2}}$$

frequency of vortex shedding $\quad n = \frac{U-u}{\ell}$

drag coefficient $\quad C_D = \frac{n \rho \Gamma h}{\frac{1}{2} \rho U^2 d} = 1.59 \frac{u}{U}\left(1 - \frac{u}{U}\right) \frac{\ell}{d}$

Strouhal number $\quad S = \frac{nd}{U}$



Boundary layer equations

Prandtl's concept of boundary layer for large Reynolds numbers : region of intense vorticity where viscous forces are of comparable magnitude to the inertial forces (Ludwig Prandtl 1904).

- Two-dimensional flow along a plane wall.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\underline{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}\right),\qquad \text{N.S. equations}$$
$$\underline{\frac{\partial v}{\partial t}}+\underline{u\frac{\partial v}{\partial x}}+\underline{v\frac{\partial v}{\partial y}}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\underline{\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)}.$$

$R=\frac{U_0\ell}{\nu}$: U_0, ℓ : representative streamwise velocity and length;

boundary layer thickness $\delta=\ell\times O(R^{-\frac{1}{2}})$,

$u=U_0\times O(1)$, $v=U_0\times O(R^{-\frac{1}{2}})$,

$$\nu\frac{\partial^2 u}{\partial x^2}\Big/\nu\frac{\partial^2 u}{\partial y^2}=O(R^{-1}),\quad \frac{1}{\rho}\frac{\partial p}{\partial y}\Big/\frac{U_0^2}{\ell}=O(R^{-\frac{1}{2}}),\quad \frac{\Delta p}{\rho U_0^2}=O(R^{-1}).$$

The equations simplify to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2},\qquad \text{Boundary layer equations}$$
$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y}.$$

At the wall $y=0$: $u=0$, $v=0$.

At the outer edge of the boundary layer $y=\delta$: $u=U$;

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}.$$

Thus, U or p is regarded as prescribed function of x and t.

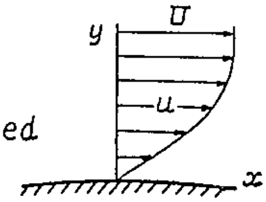
- Derivation of boundary layer equations in stretched coordinates.

$$\hat{x}=\frac{x}{\ell},\quad \hat{y}=R^{\frac{1}{2}}\frac{y}{\ell},\quad \hat{u}=\frac{u}{U_0},\quad \hat{v}=R^{\frac{1}{2}}\frac{v}{U_0},\quad \hat{p}=\frac{p-p_0}{\rho U_0^2},\quad \hat{t}=\frac{U_0 t}{\ell}:$$
$$\frac{\partial \hat{u}}{\partial \hat{x}}+\frac{\partial \hat{v}}{\partial \hat{y}}=0,$$
$$\frac{\partial \hat{u}}{\partial \hat{t}}+\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{u}}{\partial \hat{y}}=-\frac{\partial \hat{p}}{\partial \hat{x}}+\underline{R^{-1}\frac{\partial^2 \hat{u}}{\partial \hat{x}^2}}+\frac{\partial^2 \hat{u}}{\partial \hat{y}^2},$$
$$\underline{R^{-1}\left(\frac{\partial \hat{v}}{\partial \hat{t}}+\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{v}}{\partial \hat{y}}\right)}=-\frac{\partial \hat{p}}{\partial \hat{y}}+\underline{R^{-2}\frac{\partial^2 \hat{v}}{\partial \hat{x}^2}}+\underline{R^{-1}\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}}.$$

It is seen that the boundary layer equations correspond to the limiting form of the Navier-Stokes equations as the Reynolds number tends to infinity.

- Curved wall. The boundary layer equations differ only in

$$-\kappa u^2 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \ ; \qquad \frac{\Delta p}{\rho U_0^2} = O(R^{-\frac{1}{2}}),$$



κ being the curvature of the solid wall. Thus, the equations derived for a plane wall are applied to the flow along a curved wall provided that (x, y) is now regarded as a system of curvilinear coordinates given by the solid wall and parallel curves and the normals to the wall, orthogonal at the wall, but not necessarily elsewhere. Necessary restrictions on the curvature are that $\kappa \ell = O(1)$, $(d\kappa/dx)\,\ell^2 = O(1)$.

- Formulation in terms of prescribed velocity $U(x,t)$:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \ ;$$

$$y = 0 : \quad u = 0, \quad v = 0 \ ;$$

$$y \to \infty : \quad u \to U(x,t).$$

- Flow over a semi-infinite plate : $U(x,t) = U_0 = \text{constant}.$

$$\psi = (2\nu x U_0)^{\frac{1}{2}} f(\eta), \qquad \eta = \left(\frac{U_0}{2\nu x}\right)^{\frac{1}{2}} y \ ;$$

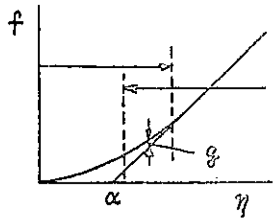
$$u = \frac{\partial \psi}{\partial y} = U_0 f', \quad v = -\frac{\partial \psi}{\partial x} = \left(\frac{\nu U_0}{2x}\right)^{\frac{1}{2}}(\eta f' - f),$$

$$\nu\frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = \frac{U_0^2}{2x}\left\{ f''' + \eta f' f'' - (\eta f' - f) f'' \right\} = 0 \ ;$$

$$f''' + f f'' = 0, \qquad f(0) = 0, \ f'(0) = 0, \ f'(\infty) = 1.$$

For small η : $f = \frac{a}{2!}\eta^2 - \frac{a^2}{5!}\eta^5 + \frac{11a^3}{8!}\eta^8 - \frac{375a^4}{11!}\eta^{11} + \cdots, \quad a = f''(0).$

For large η : $f = \eta - \alpha + \beta \int_\infty^\eta d\eta \int_\infty^\eta e^{-\frac{1}{2}(\eta-\alpha)^2} d\eta$



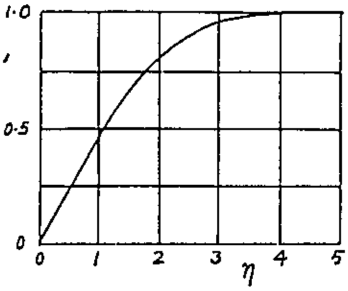
matching $a = 0{\cdot}470, \quad \alpha = 1{\cdot}224, \quad \beta = 0{\cdot}327.$

more accu. $a = 0{\cdot}46960, \ \alpha = 1{\cdot}21678, \ \beta = 0{\cdot}3305.$

boundary layer thickness δ_{995} :

$$f' = \frac{u}{U_0} = 0.995 \quad \text{at} \quad \eta = \left(\frac{U_0}{2\nu x}\right)^{\frac{1}{2}} y = 3.727,$$

$$\delta_{995} = 3.727\left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}} = 5.271\left(\frac{\nu x}{U_0}\right)^{\frac{1}{2}}.$$



displacement thickness δ^* :

$$\int_0^{y_1} u\,dy = U_0(y_1 - \delta^*); \quad \delta^* = \int_0^{y_1}\left(1 - \frac{u}{U_0}\right)dy$$

$$= \left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}} \int_0^{\eta_1} (1 - f')\,d\eta = \left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}} \{\eta_1 - f(\eta_1)\} = \alpha\left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}}$$

$$= 1.21678\left(\frac{2\nu x}{U_0}\right)^{\frac{1}{2}} = 1.721\left(\frac{\nu x}{U_0}\right)^{\frac{1}{2}} = \frac{\delta_{995}}{3.063}.$$

normal velocity $v_1 = \left(\frac{\nu U_0}{2x}\right)^{\frac{1}{2}} (\eta f' - f)_{y=y_1} = \alpha\left(\frac{\nu U_0}{2x}\right)^{\frac{1}{2}} = U_0 \frac{d\delta^*}{dx}.$

wall shear stress $\tau_0 = \rho\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} = a\,\rho U_0\left(\frac{\nu U_0}{2x}\right)^{\frac{1}{2}} = 0.332\,\rho U_0^2\left(\frac{\nu}{U_0 x}\right)^{\frac{1}{2}}.$

frictional drag coefficient $C_f = \dfrac{\int_0^{\ell} \tau_0 dx}{\frac{1}{2}\rho U_0^2 \ell} = 1.328\left(\dfrac{U_0 \ell}{\nu}\right)^{-\frac{1}{2}}.$

- Series solution for $U(x) = cx(1 + b_2 x^2 + b_4 x^4 + b_6 x^6 + \cdots)$:

$$\psi = (c\nu)^{\frac{1}{2}} x \Big[f_0(\eta) + b_2 f_2(\eta) x^2 + \{ b_4 f_4(\eta) + b_2^2 f_{22}(\eta) \} x^4$$
$$+ \{ b_6 f_6(\eta) + b_4 b_2 f_{42}(\eta) + b_2^3 f_{222}(\eta) \} x^6 + \cdots \Big];$$

$$\eta = \left(\frac{c}{\nu}\right)^{\frac{1}{2}} y; \quad f_0''' + f_0 f_0'' + 1 - f_0'^2 = 0, \quad f_0(0) = f_0'(0) = 0, \quad f_0'(\infty) = 1,$$

$$f_2''' + f_0 f_2'' - 4 f_0' f_2' + 3 f_0'' f_2 = -4, \quad f_2(0) = f_2'(0) = 0, \quad f_2'(\infty) = 1,$$

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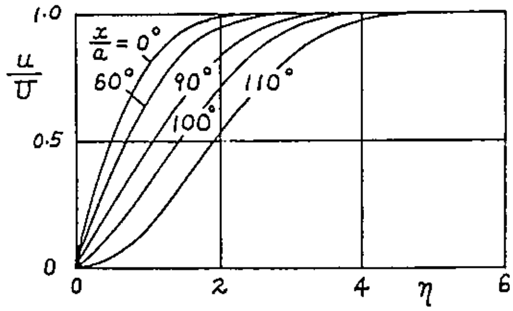
Example. Potential flow past a circular cylinder of radius a.

$$U(x) = 2U_0 \sin\frac{x}{a} = \frac{2U_0}{a} x \left(1 - \frac{1}{6}\frac{x^2}{a^2} + \frac{1}{120}\frac{x^4}{a^4} - \frac{1}{5040}\frac{x^6}{a^6} + \frac{1}{362880}\frac{x^8}{a^8} - \cdots\right)$$



Retaining terms up to $b_8 x^8$ predicts $\tau_0 = \rho\nu(\partial u/\partial y)_{y=0}$ to vanish at $x/a = 109.6°$, which is far from the most accurate numerical result $104.45°$.

Momentum and energy integral relations for steady two-dimensional boundary layer are obtained by direct integration of the momentum equation $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$ with respect to y across the boundary layer and by integration after multiplying by u, respectively:

$$\frac{d}{dx}\int_0^\delta u^2 dy - U\frac{d}{dx}\int_0^\delta u\,dy = -\frac{\delta}{\rho}\frac{dp}{dx} - \nu\left(\frac{\partial u}{\partial y}\right)_{y=0},$$

$$\frac{1}{2}\frac{d}{dx}\int_0^\delta u^3 dy - \frac{1}{2}U^2\frac{d}{dx}\int_0^\delta u\,dy = \frac{1}{2}\frac{dU^2}{dx}\int_0^\delta u\,dy - \nu\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy.$$

It is convenient to write the relations in the form

$$\frac{d}{dx}(U^2\theta) = \frac{\delta^*}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial u}{\partial y}\right)_{y=0},$$

$$\frac{1}{2}\frac{d}{dx}(U^3\theta^*) = \nu\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy$$

in terms of the displacement thickness δ^*, momentum thickness θ, and energy thickness θ^*, defined by

$$U\delta^* = \int_0^\delta (U-u)\,dy,\quad U^2\theta = \int_0^\delta u(U-u)\,dy,\quad U^3\theta^* = \int_0^\delta u(U^2-u^2)\,dy.$$

One-parameter family of velocity profiles

With the velocity profile of the form

$$\frac{u}{U} = 1-\left(1-\frac{y}{\delta}\right)^{m-1}\left\{1+(m-1-\alpha)\frac{y}{\delta}\right\},$$

it is possible to satisfy the condition that $u=0$ at $y=0$ and $\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \cdots = \frac{\partial^{m-2}u}{\partial y^{m-2}} = 0$ at $y=\delta$. For $m=4$, we have

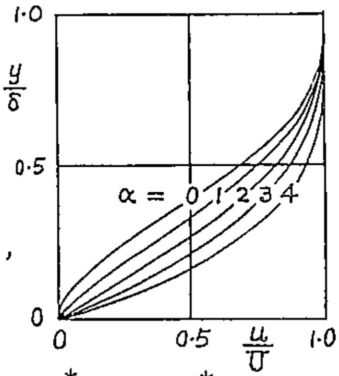
$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \alpha\frac{U}{\delta},$$

$$\delta^* = \frac{\delta}{20}(8-\alpha),$$

$$\theta = \frac{\delta}{1260}(144+12\alpha-5\alpha^2),$$

$$\theta^* = \frac{\delta}{60060}(10512+876\alpha-253\alpha^2-21\alpha^3),$$

$$\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy = \frac{U^2}{35\delta}(48-4\alpha+3\alpha^2).$$



Consequently, non-dimensional quantities $H=\frac{\delta^*}{\theta}$, $G=\frac{\theta^*}{\theta}$, $P=\frac{2\theta}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}$ and $Q=\frac{4\theta^*}{U^2}\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy$ are determined as functions of α:

$$G = \frac{3}{143}\,\frac{10512 + 876\alpha - 253\alpha^2 - 21\alpha^3}{144 + 12\alpha - 5\alpha^2}, \quad H = \frac{63(8-\alpha)}{144 + 12\alpha - 5\alpha^2},$$

$$P = \frac{\alpha}{630}\,(144 + 12\alpha - 5\alpha^2),$$

$$Q = \frac{1}{525525}\,(48 - 4\alpha + 3\alpha^2)(10512 + 876\alpha - 253\alpha^2 - 21\alpha^3).$$

α	G	$\ln G$	H	P	Q	$P - Q/G^2$
0	1.531	0.4262	3.500	0	0.960	−0.4094
0.25	1.532	0.4268	3.328	0.0582	0.962	−0.3516
0.50	1.535	0.4285	3.176	0.1181	0.968	−0.2928
0.75	1.539	0.4311	3.041	0.1788	0.979	−0.2345
1.00	1.544	0.4345	2.921	0.2397	0.994	−0.1772
1.25	1.550	0.4383	2.813	0.3000	1.014	−0.1219
1.50	1.557	0.4425	2.716	0.3589	1.038	−0.0693
1.75	1.564	0.4471	2.630	0.4158	1.066	−0.0199
2.00	1.571	0.4518	2.554	0.4698	1.097	0.0255
2.25	1.579	0.4566	2.486	0.5203	1.130	0.0667
2.50	1.586	0.4613	2.427	0.5665	1.165	0.1033
3.00	1.600	0.4700	2.333	0.6429	1.234	0.1603
3.50	1.611	0.4767	2.273	0.6931	1.290	0.1961
4.00	1.615	0.4796	2.250	0.7111	1.313	0.2030

Method of solution

Momentum and energy integral relations are

$$\frac{U}{\nu}\frac{d\theta^2}{dx} + 2(2+H)\frac{\theta^2}{\nu}\frac{dU}{dx} = P, \qquad (1)$$

$$\frac{U}{\nu}\frac{dG^2\theta^2}{dx} + 6\,\frac{G^2\theta^2}{\nu}\frac{dU}{dx} = Q. \qquad (2)$$

Put $\lambda = \frac{\theta^2}{\nu}\frac{dU}{dx}$ and eliminate $\frac{d\theta^2}{dx}$:

$$\lambda(H-1) = \frac{1}{2}\left(P - \frac{Q}{G^2}\right) + \lambda\,\frac{d\ln G}{d\ln U}. \qquad (3)$$

Flat plate : $\lambda = 0$, $P - \frac{Q}{G^2} = 0$, $\alpha = 1{\cdot}857$;

$P = 0{\cdot}439$, $Q = 1{\cdot}079$, $H = 2{\cdot}60$, $G = 1{\cdot}567$.

(exact) $0{\cdot}441$, $1{\cdot}090$, $2{\cdot}59$, $1{\cdot}572$.

Stagnation point : $U = 0$, $\frac{Q}{3G^2} - \frac{P}{2+H} = 0$;

$0{\cdot}1677 - 0{\cdot}1673 = 0{\cdot}0004$ at $\alpha = 4$.

$P = 0{\cdot}711$ (exact $0{\cdot}721$), $\lambda = 0{\cdot}084$ (exact $0{\cdot}085$)

Assume $G = G_m$, $Q = Q_m$ in (2) : $\frac{\theta^2}{\nu} = \frac{Q_m}{G_m^2}\,U^{-6}\int_0^x U^5\,dx$.

$G_m = 1{\cdot}567$, $Q_m = 1{\cdot}079$: $\frac{\theta^2}{\nu} = 0{\cdot}439\,U^{-6}\int_0^x U^5\,dx$.

Quadrature formula : Hudimoto, Tani, Walz (1940), Thwaites (1949).

Successive approximations:

$$\frac{\theta^2}{\nu} = 0.439\, U^{-6}\int_0^x U^5 dx\,;$$

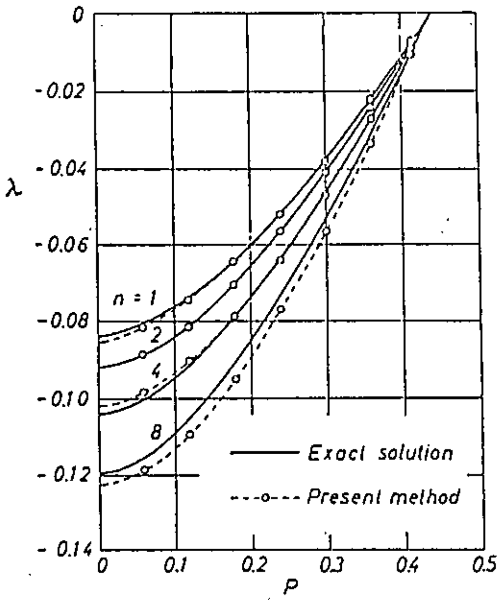
$\lambda = \frac{\theta^2}{\nu}\frac{dU}{dx}$ is determined as a function of x;

α is then determined as a function of x by (3);

$G = G_m'$, $Q = Q_m' + r\lambda$:

$$\frac{\theta^2}{\nu} = \frac{Q_m'}{G^2} U^{-(6-r)}\int_0^x U^{5-r} dx.$$

I. Tani, J. Aero. Sci. <u>21</u>, 487, 1954.



Simplification and improvement:

$$G = 1.531 + 0.0367\,P + 0.1310\,P^2,$$

$$H = 3.500 - 2.502\,P + 1.000\,P^2,$$

$$Q = 2G(0.3135 - 0.0314\,P + 0.2425\,P^2).$$

These relations are exact at $P = 0.4411$ (flat plate) and $P = 0.7207$ (stagnation point). I. Tani & N. J. Yu, Recent Research on Unsteady Boundary Layers (Ed. Eichelbrenner), 886, Quebec 1972.

<u>Occurrence of reversed flows</u>

With the quadrature formula $\theta^2/\nu = 0.441\, U^{-6}\int_0^x U^5 dx$, the first term of the left side of the momentum integral relation

$$\frac{d}{dx}(U^2\theta) + U\delta^*\frac{dU}{dx} = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

is written in the form $(\nu U/\theta)\{0.220 - (\theta^2/\nu)(dU/dx)\}$, which turns out to be positive, having a minimum value of 0.135 at the stagnation point. For accelerated flows the second term is positive, making it necessary for the right side to be always positive. For decelerated flows, however, the second term is negative, so that there is a possibility for the right side to be negative, particularly when the second term of the left side is excessively negative. Thus, the situation is brought about where the wall shear stress vanishes and the reversed flow sets in, provided the velocity U decreases (or the pressure p increases) in the streamwise direction.

Instability of laminar flows

- Method of small disturbances.

  A small-amplitude disturbance of appropriate type is superposed on a steady-state solution of the Navier-Stokes equations. The flow is considered stable if the disturbance ultimately decays to zero, but is unstable if the disturbance ensues that is permanently different from zero. Instability does not lead directly to turbulent flow.

- Small disturbance superposed on a steady bounded parallel flow.

$$u_\alpha^* = U_\alpha(x_2) + u_\alpha(x_\alpha, t), \quad p^* = P(x_1) + p(x_\alpha, t):$$

$$U_1 = U_0 F\left(\frac{x_2}{\ell}\right), \quad U_2 = U_3 = 0, \quad \frac{1}{\rho}\frac{dP}{dx_1} = \nu \frac{d^2 U_1}{dx_2^2} ;$$

$$u_\alpha \ll U_0, \quad p \ll \rho U_0^2, \quad \alpha = 1, 2, 3.$$

$$\frac{\partial u_\alpha^*}{\partial x_\alpha} = 0, \quad \frac{\partial u_\alpha^*}{\partial t} + u_\beta^* \frac{\partial u_\alpha^*}{\partial x_\beta} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x_\alpha} + \nu \frac{\partial^2 u_\alpha^*}{\partial x_\beta \partial x_\beta} ;$$

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0, \quad \frac{\partial u_\alpha}{\partial t} + U_1 \frac{\partial u_\alpha}{\partial x_1} + \delta_{\alpha 1}\frac{dU_1}{dx_2} u_2 = -\frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \nu \frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta}$$

- Assumption for small disturbance.

$$\xi = \frac{x_1}{\ell}, \quad \eta = \frac{x_2}{\ell}, \quad \zeta = \frac{x_3}{\ell}, \quad \tau = \frac{U_0 t}{\ell}, \quad R = \frac{U_0 \ell}{\nu} ;$$

$$u^+ = \frac{u_1}{U_0} = \hat{u}(\eta)\exp\{i\kappa(\xi - c\tau) + i\lambda\zeta\},$$

$$v^+ = \frac{u_2}{U_0} = \hat{v}(\eta)\exp\{i\kappa(\xi - c\tau) + i\lambda\zeta\},$$

$$w^+ = \frac{u_3}{U_0} = \hat{w}(\eta)\exp\{i\kappa(\xi - c\tau) + i\lambda\zeta\},$$

$$p^+ = \frac{p}{\rho U_0^2} = \hat{p}(\eta)\exp\{i\kappa(\xi - c\tau) + i\lambda\zeta\}.$$

κ and λ : wave number in x_1 and x_3 directions,

$c = c_r + ic_i$, c_r : phase speed, c_i : amplification rate,

$c_i < 0$ stable, $c_i > 0$ unstable.

$$\left.\begin{aligned} & i(\kappa\hat{u} + \lambda\hat{w}) + \frac{d\hat{v}}{d\eta} = 0, \\ & \left\{\frac{1}{R}\left(\frac{d^2}{d\eta^2} - \kappa^2 - \lambda^2\right) - i\kappa(F - c)\right\} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{bmatrix} = \begin{bmatrix} \frac{dF}{d\eta}\hat{v} + i\kappa\hat{p} \\ \frac{d\hat{p}}{d\eta} \\ i\lambda\hat{p} \end{bmatrix} \end{aligned}\right\}$$

• Two-dimensional disturbance. $\hat{w} = 0$, $\lambda = 0$.

$$\left.\begin{aligned} & i\kappa\hat{u} + \frac{d\hat{v}}{d\eta} = 0, \\ & \left\{\frac{1}{R}\left(\frac{d^2}{d\eta^2} - \kappa^2\right) - i\kappa(F-c)\right\}\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix} = \begin{bmatrix}\frac{dF}{d\eta}\hat{v} + i\kappa\hat{p}\\ \frac{d\hat{p}}{d\eta}\end{bmatrix}\end{aligned}\right\}.$$

On the other hand, application of the transformation

$$\tilde{\kappa}\tilde{u} = \kappa\hat{u} + \lambda\hat{w}, \quad \tilde{v} = \hat{v}, \quad \tilde{\kappa}\tilde{p} = \kappa\hat{p},$$
$$\tilde{\kappa}^2 = \kappa^2 + \lambda^2, \quad \tilde{c} = c, \quad \tilde{\kappa}\tilde{R} = \kappa R$$

brings the set of equations to the form

$$\left.\begin{aligned} & i\tilde{\kappa}\tilde{u} + \frac{d\tilde{v}}{d\eta} = 0, \\ & \left\{\frac{1}{\tilde{R}}\left(\frac{d^2}{d\eta^2} - \tilde{\kappa}^2\right) - i\tilde{\kappa}(F-\tilde{c})\right\}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix} = \begin{bmatrix}\frac{dF}{d\eta}\tilde{v} + i\tilde{\kappa}\tilde{p}\\ \frac{d\tilde{p}}{d\eta}\end{bmatrix}\end{aligned}\right\},$$

which indicates that the disturbance characteristics marked with twiddle represent those of the equivalent two-dimensional disturbance. Since $\tilde{\kappa} > \kappa$, we have $\tilde{R} < R$, implying that the problem of three-dimensional instability at a Reynolds number R is equivalent to a problem of two-dimensional instability at a lower Reynolds number $\tilde{R}$. As regards the neutral disturbance, therefore, the minimum critical Reynolds number (above which instability sets in) occurs with a two-dimensional disturbance (Squire theorem).

• Orr-Sommerfeld equation for two-dimensional disturbance.

Introducing stream function by $\hat{u} = \frac{d\Psi}{d\eta}$, $\hat{v} = -i\kappa\Psi$, and eliminating $\hat{p}$ by differentiation, we get the differential equation

$$\left\{(F-c)\left(\frac{d^2}{d\eta^2} - \kappa^2\right) - \frac{d^2F}{d\eta^2} - \frac{1}{i\kappa R}\left(\frac{d^4}{d\eta^4} - 2\kappa^2\frac{d^2}{d\eta^2} + \kappa^4\right)\right\}\Psi = 0,$$

which is called Orr-Sommerfeld equation. The boundary conditions for Ψ are $\Psi = 0$, $\frac{d\Psi}{d\eta} = 0$ at $\eta = \pm 1$. With the solution in the form

$\Psi = C_1\Psi_1 + C_2\Psi_2 + C_3\Psi_3 + C_4\Psi_4$ and the abbreviation $(')$ for $d/d\eta$, the homogeneous boundary conditions lead to

$$\begin{vmatrix} \Psi_1(-1) & \Psi_2(-1) & \Psi_3(-1) & \Psi_4(-1) \\ \Psi_1(+1) & \Psi_2(+1) & \Psi_3(+1) & \Psi_4(+1) \\ \Psi_1'(-1) & \Psi_2'(-1) & \Psi_3'(-1) & \Psi_4'(-1) \\ \Psi_1'(+1) & \Psi_2'(+1) & \Psi_3'(+1) & \Psi_4'(+1) \end{vmatrix} = 0,$$

which provides an eigenvalue relation $c = c(\kappa, R\,;\, F, i)$.

- *Inviscid solution (Rayleigh).*

$$R \to \infty,\ \kappa R \to \infty : \quad (F-c)(\Psi'' - \kappa^2\Psi) - F''\Psi = 0\,, \quad (') = \frac{d}{d\eta}.$$

Boundary conditions : $\Psi(-1) = \Psi(+1) = 0$.

or : $\Psi(-1) = 0$, $\Psi(0)$ or $\Psi'(0) = 0$.

Multiply $\Psi'' - \kappa^2\Psi - \dfrac{F''\Psi}{F-c} = 0$ by complex conjugate $\tilde{\Psi}$ and integrate between $\eta = -1$ and 0 : $\int \{ |\Psi'|^2 + \kappa^2|\Psi|^2 \} d\eta + \int \dfrac{F''|\Psi|^2}{F-c} d\eta = 0$,

$$\int \{ |\Psi'|^2 + \kappa^2|\Psi|^2 \} d\eta + \int \frac{F''(F-c_r)}{|F-c|^2} |\Psi|^2 d\eta + i c_i \int \frac{F''}{|F-c|} |\Psi|^2 d\eta = 0$$

For amplified disturbances ($c_i > 0$), the vanishing of the imaginary part necessitates F'' to change its sign within the range of integration. It follows that the existence of a point of inflection in the velocity profile is a necessary condition for instability at the inviscid limit. This is also a sufficient condition for symmetrical and boundary-layer type velocity profiles.

It is to be noticed that the amplified and damped solutions are no longer complex conjugates at the inviscid limit. We take the view that the inviscid damped solution does not hold in the region considered.

- Energy balance. The equations for two-dimensional disturbance are

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \quad \frac{\partial u_1}{\partial t} + U_1 \frac{\partial u_1}{\partial x_1} + \frac{dU_1}{dx_2} u_2 + \chi_1 = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\nabla^2 u_1\,,$$

$$\frac{\partial u_2}{\partial t} + U_1 \frac{\partial u_2}{\partial x_1} + \chi_2 = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \nu\nabla^2 u_2\,,$$

where nonlinear terms

$$\chi_1 = u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} - \overline{u_1\frac{\partial u_1}{\partial x_1}} - \overline{u_2\frac{\partial u_1}{\partial x_2}},\quad \chi_2 = u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} - \overline{u_1\frac{\partial u_2}{\partial x_1}} - \overline{u_2\frac{\partial u_2}{\partial x_2}}$$

are retained. A bar denotes the mean value, the average being taken conveniently with respect to x_1. Quantities u_1, u_2, p have zero means.

Multiply the equations by u_1 and u_2 respectively, add, and integrate over a domain bounded by one wavelength $2\pi\ell/\kappa$, and by the planes $x_2 = \pm\ell$. We then obtain

$$\frac{d}{dt}\iint \frac{\rho}{2}(u_1^2+u_2^2)\,dx_1dx_2 = \iint(-\rho\overline{u_1u_2})\frac{d\overline{U}_1}{dx_2}\,dx_1dx_2 - \rho\nu\iint\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)^2 dx_1dx_2,$$

the nonlinear terms disappearing in the process of integration. The left side represents the time rate of change in kinetic energy of the disturbance, while the right side represents the energy transfer by the action of Reynolds stress $-\rho\overline{u_1u_2}$ and the rate of energy dissipation by viscosity. If the disturbance is to grow in amplitude, the first term must be positive, and there must be a dominant region of flow in which the Reynolds stress is of the same sign as the mean velocity gradient. Introduction of non-dimensional quantities gives

$$\frac{d}{d\tau}\iint \frac{1}{2}(u^{+2}+v^{+2})\,d\xi\,d\eta = \iint(-\overline{u^+v^+})\frac{dF}{d\eta}\,d\xi\,d\eta - \frac{1}{R}\iint\left(\frac{\partial v^+}{\partial\xi}-\frac{\partial u^+}{\partial\eta}\right)^2 d\xi\,d\eta,$$

which suggests the existence of a critical Reynolds number.

- Calculation of Reynolds stress.

$$u^+ = \hat{u}\,e^{i\kappa(\xi-c\tau)} = (\Psi_r' + i\Psi_i')(\cos\kappa\theta + i\sin\kappa\theta)e^{\kappa c_i\tau} \qquad (\theta = \xi - c_r\tau)$$
$$= (\Psi_r'\cos\kappa\theta - \Psi_i'\sin\kappa\theta)e^{\kappa c_i\tau},$$
$$v^+ = \hat{v}\,e^{i\kappa(\xi-c\tau)} = -i\kappa(\Psi_r + i\Psi_i)(\cos\kappa\theta + i\sin\kappa\theta)e^{\kappa c_i\tau}$$
$$= \kappa(\Psi_r\sin\kappa\theta + \Psi_i\cos\kappa\theta)e^{\kappa c_i\tau}.$$

$$-u^+v^+ = \kappa\left\{\Psi_r\Psi_i'\sin^2\kappa\theta - \Psi_i\Psi_r'\cos^2\kappa\theta - (\Psi_r\Psi_r' - \Psi_i\Psi_i')\sin\kappa\theta\cos\kappa\theta\right\}e^{2\kappa c_i\tau},$$
$$-\overline{u^+v^+} = \frac{\kappa}{2\pi}\int_0^{\frac{2\pi}{\kappa}}(-u^+v^+)\,d\xi = \frac{\kappa}{2\pi}\int_0^{\frac{2\pi}{\kappa}}(-u^+v^+)\,d\theta = \frac{\kappa}{2}(\Psi_r\Psi_i' - \Psi_i\Psi_r')\,e^{2\kappa c_i\tau}$$
$$= \frac{i\kappa}{4}(\Psi\tilde{\Psi}' - \tilde{\Psi}\Psi')\,e^{2\kappa c_i\tau};$$

Similarly $\overline{u^{+2}} = \frac{1}{2}\Psi'\tilde{\Psi}'e^{2\kappa c_i\tau}$, $\quad \overline{v^{+2}} = \frac{\kappa^2}{2}\Psi\tilde{\Psi}\,e^{2\kappa c_i\tau}$.

For inviscid disturbances

$$\Psi\left(\tilde{\Psi}''-\kappa^2\tilde{\Psi}-\frac{F''\tilde{\Psi}}{F-\tilde{c}}\right)-\tilde{\Psi}\left(\Psi''-\kappa^2\Psi-\frac{F''\Psi}{F-c}\right)=0,$$

$$\Psi\tilde{\Psi}''-\tilde{\Psi}\Psi''+F''\Psi\tilde{\Psi}\left(\frac{1}{F-c}-\frac{1}{F-\tilde{c}}\right)=0,$$

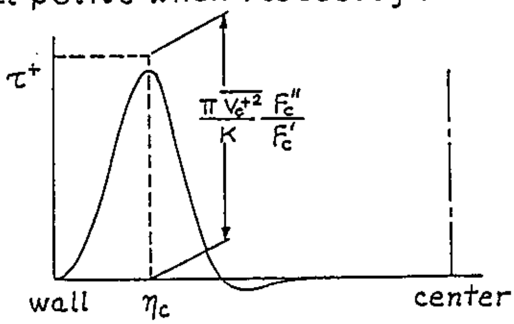
$$\frac{d}{d\eta}\left(\Psi\tilde{\Psi}'-\tilde{\Psi}\Psi'\right)+\frac{2ic_iF''\Psi\tilde{\Psi}}{|F-c|^2}=0,$$

$$\frac{d}{d\eta}\left(-\overline{u^+v^+}\right)=\frac{\overline{v^{+2}}}{\kappa}\frac{c_iF''}{(F-c_r)^2+c_i^2}.$$

For neutral disturbances ($c_i=0$), therefore, the non-dimensional Reynolds stress $\tau^+\equiv-\overline{u^+v^+}$ is constant, except possibly at the critical point $\eta=\eta_c$ where $F=c_r$. Integration of $d\tau^+/d\eta$ from η_c-0 to η_c+0 yields

$$\tau^+(\eta_c+0)-\tau^+(\eta_c-0)=\int_{F(\eta_c-0)}^{F(\eta_c+0)}\frac{\overline{v^{+2}}}{\kappa}\frac{F''}{F'}\frac{c_i}{(F-c_r)^2+c_i^2}\,dF$$

which tends to $\left.\frac{\pi\overline{v^{+2}}}{\kappa}\frac{F''}{F'}\right|_{\eta=\eta_c}$ in the limit $c_i\to 0$ by virtue of the well-known relation* in potential theory connecting the field strength with the strength of source distributed on the line $c_i=0$. Thus, the discontinuity of τ^+ at $\eta=\eta_c$ would be $(\pi\overline{v_c^{+2}}/\kappa)(F_c''/F_c')$ for neutral disturbances. Since τ^+ must vanish on the solid wall as well as on the centerline, τ^+ would attain a maximum in the neighborhood of the critical point when viscosity is taken into account.



$$*\ \lim_{y\to 0}\int_{\xi_1}^{\xi_2}\frac{f(\xi)\,y\,d\xi}{(x-\xi)^2+y^2}=\pi f(x)$$

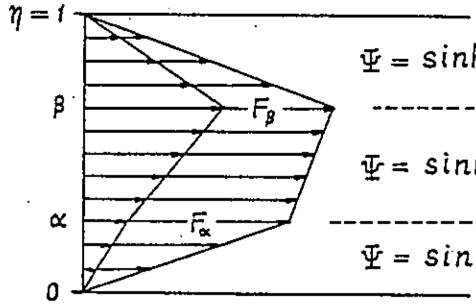
Inviscid analysis of channel flow with piecewise linear profile

$$R\to\infty,\quad \kappa c=\omega:\qquad \left(F-\frac{\omega}{\kappa}\right)(\Psi''-\kappa^2\Psi)-F''\Psi=0,$$

$$\left[\left(F-\frac{\omega}{\kappa}\right)\Psi'-F'\Psi\right]'-\kappa^2\left(F-\frac{\omega}{\kappa}\right)\Psi=0.$$

$$F''=0:\quad \Psi''-\kappa^2\Psi=0,\quad \Psi=\sinh\kappa\eta,\ \cosh\kappa\eta.$$

$$\text{matching conditions:}\quad \Delta\Psi=0,\quad \left(F-\frac{\omega}{\kappa}\right)\Delta\Psi'-\Psi\Delta F'=0.$$



$$\Psi=\sinh\kappa\eta+A\sinh\kappa(\eta-\alpha)+B\sinh\kappa(\eta-\beta)$$

$$D_\beta=(\Delta F')_{\eta=\beta}=-\frac{F_\beta}{1-\beta}-\frac{F_\beta-F_\alpha}{\beta-\alpha}$$

$$\Psi=\sinh\kappa\eta+A\sinh\kappa(\eta-\alpha)$$

$$D_\alpha=(\Delta F')_{\eta=\alpha}=\frac{F_\beta-F_\alpha}{\beta-\alpha}-\frac{F_\alpha}{\alpha}$$

$$\Psi=\sinh\kappa\eta$$

$$\text{matching at }\eta=\alpha:\quad \left(F_\alpha-\frac{\omega}{\kappa}\right)\kappa A-D_\alpha\sinh\kappa\alpha=0,$$

$$\text{matching at }\eta=\beta:\quad \left(F_\beta-\frac{\omega}{\kappa}\right)\kappa B-D_\beta\{\sinh\kappa\beta+A\sinh\kappa(\beta-\alpha)\}=0;$$

$$\text{no-slip at }\eta=1:\quad \sinh\kappa+A\sinh\kappa(1-\alpha)+B\sinh\kappa(1-\beta)=0.$$

$$\text{eliminating A and B:}\quad \sinh\kappa+\frac{D_\alpha\sinh\kappa\alpha}{\kappa F_\alpha-\omega}\sinh\kappa(1-\alpha)$$

$$+\left\{\frac{D_\beta\sinh\kappa\beta}{\kappa F_\beta-\omega}+\frac{D_\alpha D_\beta\sinh\kappa\alpha\sinh\kappa(\beta-\alpha)}{(\kappa F_\alpha-\omega)(\kappa F_\beta-\omega)}\right\}\sinh\kappa(1-\beta)=0.$$

$$P\omega^2+Q\omega+R=0:$$

$$P=\sinh\kappa,$$

$$Q=-\kappa(F_\alpha+F_\beta)\sinh\kappa-D_\alpha\sinh\kappa\alpha\sinh\kappa(1-\alpha)-D_\beta\sinh\kappa\beta\sinh\kappa(1-\beta),$$

$$R=\kappa^2F_\alpha F_\beta\sinh\kappa+\kappa F_\beta D_\alpha\sinh\kappa\alpha\sinh\kappa(1-\alpha)+\kappa F_\alpha D_\beta\sin\kappa\beta\sinh\kappa(1-\beta)$$

$$+D_\alpha D_\beta\sinh\kappa\alpha\sinh\kappa(1-\beta)\sinh\kappa(\beta-\alpha).$$

$$Q^2-4PR=\{\kappa(F_\alpha-F_\beta)\sinh\kappa+D_\alpha\sinh\kappa\alpha\sinh\kappa(1-\alpha)-D_\beta\sinh\kappa\beta\sinh\kappa(1-\beta)\}^2$$

$$+4D_\alpha D_\beta\sinh^2\kappa\alpha\sinh^2\kappa(1-\beta).$$

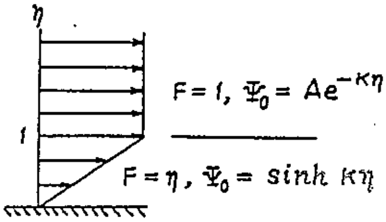
If D_α and D_β are of the same sign, so that the velocity profile is of one curvature throughout, the roots are real and therefore the disturbed motion is stable.

Correction for slip at wall (Prandtl)

$$\left(F-\frac{\omega}{\kappa}\right)(\Psi''-\kappa^2\Psi)-F''\Psi-\frac{1}{i\kappa R}\left(\Psi''''-2\kappa^2\Psi''+\kappa^4\Psi\right)=0, \qquad c=\frac{\omega}{\kappa}.$$

$\Psi = \Psi_0 + \Psi_1$: inviscid solution + viscous correction.

piecewise linear profile : $F''=0$.



- Inviscid solution $\Psi_0''-\kappa^2\Psi_0=0$: $\Psi_0 = e^{\pm\kappa\eta}$.

$\Delta\Psi_0 = 0$: $\quad Ae^{-\kappa}-\underline{\sinh\kappa}=0$;

$\left(F-\frac{\omega}{\kappa}\right)\Delta\Psi_0'-\Psi_0\Delta F'=0$:

$$\left(1-\frac{\omega}{\kappa}\right)(-\kappa\underline{Ae^{-\kappa}}-\kappa\cosh\kappa)-\underline{\sinh\kappa}\,(0-1)=0.$$

$$(\omega-\kappa)(\underline{\sinh\kappa}+\cosh\kappa)+\underline{\sinh\kappa}=0,$$

$$\omega=\omega_0=\kappa-\tfrac{1}{2}(1-e^{-2\kappa}). \qquad \Psi_0'=\kappa\cosh\kappa\eta\neq 0 \ \text{ at } \eta=0.$$

- Viscous correction close to wall.

$$F\ll\frac{\omega}{\kappa}:\ -\frac{\omega}{\kappa}\Psi_1''-\frac{1}{i\kappa R}\Psi_1''''=0;\quad \eta=\varepsilon Y:\ \frac{d^4\Psi_1}{dY^4}+i\omega R\varepsilon^2\frac{d^2\Psi_1}{dY^2}=0.$$

$$\omega R\varepsilon^2=1:\ \frac{d^4\Psi_1}{dY^4}+i\frac{d^2\Psi_1}{dY^2}=0,\quad \frac{d^2\Psi_1}{dY^2}=Be^{\frac{1-i}{\sqrt2}Y}+Ce^{-\frac{1-i}{\sqrt2}Y}=Ce^{-\frac{1-i}{\sqrt2}Y}.$$

$$\Psi_1'=\frac{1}{\varepsilon}\frac{d\Psi_1}{dY}=-\frac{\sqrt2\,C\varepsilon^{-1}}{1-i}e^{-\frac{1-i}{\sqrt2}Y},\qquad \varepsilon=\frac{1}{\sqrt{\omega R}}.$$

No slip at $\eta=0$: $\quad \Psi_0'(0)+\Psi_1'(0)=\kappa-\dfrac{\sqrt2\,C\varepsilon^{-1}}{1-i}=0, \qquad \dfrac{\sqrt2\,C\varepsilon^{-1}}{1-i}=\kappa.$

$$\Psi_1'=-\kappa e^{-\frac{1-i}{\sqrt2}Y},\quad \Psi_1=\frac{\sqrt2\,\kappa\varepsilon}{1-i}\left[e^{-\frac{1-i}{\sqrt2}Y}\right]_0^Y=-\frac{1+i}{\sqrt2}\kappa\varepsilon\left\{1-e^{-\frac{1-i}{\sqrt2}Y}\right\}.$$

$Y\to\infty$: $\quad \Psi_{1\infty}=-\dfrac{1+i}{\sqrt2}\kappa\varepsilon=-\dfrac{1+i}{\sqrt2}\dfrac{\kappa}{\sqrt{\omega R}}$, a correction to be applied to $\underline{\sinh\kappa}$.

$$(\omega-\kappa)(\underline{\sinh\kappa+\Psi_{1\infty}}+\cosh\kappa)+\underline{\sinh\kappa+\Psi_{1\infty}}=0,$$

$$\omega-\kappa=-\frac{\sinh\kappa+\Psi_{1\infty}}{e^{\kappa}+\Psi_{1\infty}}=-\tfrac12(1-e^{-2\kappa})-\tfrac12(1+e^{-4\kappa})\Psi_{1\infty}.$$

$$\omega=\omega_0+\omega_1:\qquad \omega_0=\kappa-\tfrac12(1-e^{-2\kappa}),\quad \omega_1=\frac{1+i}{2\sqrt2}\frac{\kappa}{\sqrt{\omega R}}(1+e^{-4\kappa}).$$

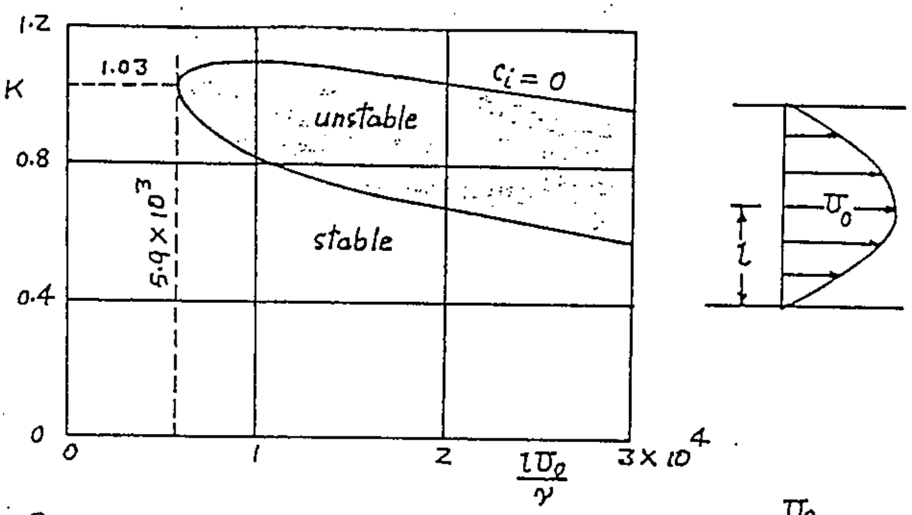
$$\omega_{1i}=\frac{1}{2\sqrt2}\frac{\kappa}{\sqrt{\omega_0R}}(1+e^{-4\kappa})>0, \text{ destabilizing.}$$

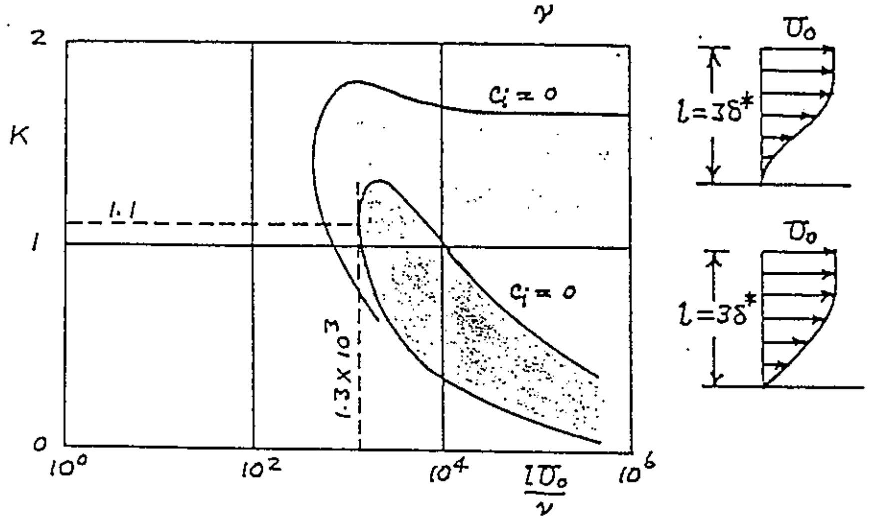
- Viscous correction for critical layer.

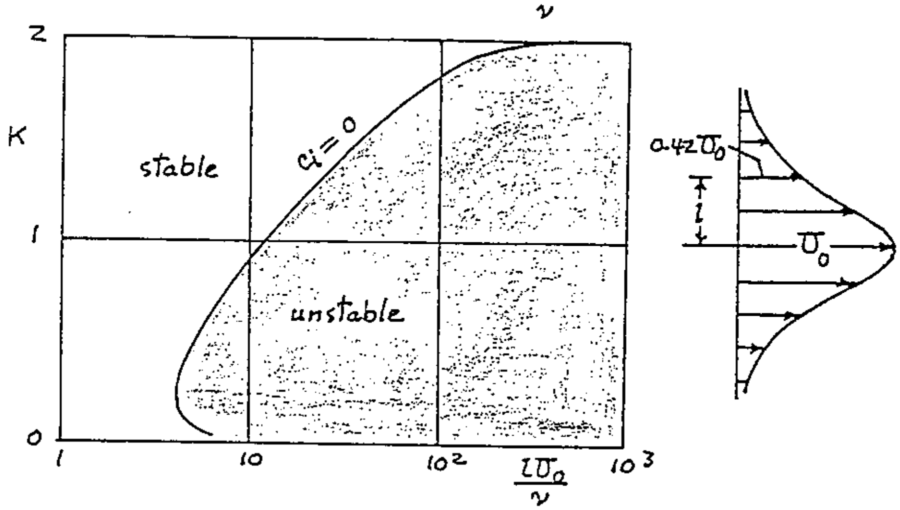
$$F-\frac{\omega}{\kappa}=F_c'(\eta-\eta_c),\quad \eta-\eta_c=\varepsilon Y,\quad F''=F_c'',\quad \varepsilon^3\kappa RF_c'=1.$$

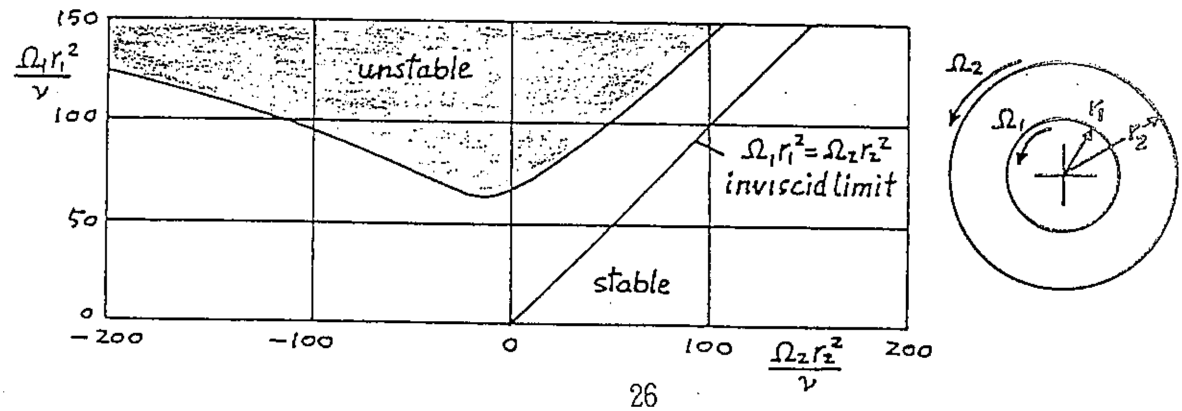
$$\left(F-\frac{\omega}{\kappa}\right)\Psi''-F''\Psi-\frac{1}{i\kappa R}\Psi''''=0,\qquad i\frac{d^4\Psi}{dY^4}+Y\frac{d^2\Psi}{dY^2}-\varepsilon\frac{F_c''}{F_c'}\Psi=0$$

$$\Psi=\Psi_0+\varepsilon\Psi_1,\quad i\frac{d^4\Psi_1}{dY^4}+Y\frac{d^2\Psi_1}{dY^2}=\frac{F_c''}{F_c}\Psi_0,\ \ldots$$


1.2
K
1.03
0.8
0.4
0
5.9×10³
unstable
stable
$c_i=0$
0
1
2
3×10⁴
$\frac{lU_0}{\nu}$
U_0
l


2
K
1.1
1
0
$c_i=0$
$c_i=0$
1.3×10³
10⁰
10²
10⁴
10⁶
$\frac{lU_0}{\nu}$
U_0
$l=3\delta^*$
U_0
$l=3\delta^*$


2
K
1
0
stable
unstable
$c_i=0$
1
10
10²
10³
$\frac{lU_0}{\nu}$
0.42U_0
l
U_0


150
$\frac{\Omega_1 r_1^2}{\nu}$
100
50
0
-200
-100
0
100
200
$\frac{\Omega_2 r_2^2}{\nu}$
unstable
stable
$\Omega_1 r_1^2=\Omega_2 r_2^2$
inviscid limit
Ω_2
Ω_1
r_1
r_2

Turbulent flows: general formulation.

Continuity and momentum equations:

$$\frac{\partial u_\alpha^*}{\partial x_\alpha} = 0, \qquad \alpha, \beta = 1, 2, 3. \tag{1}$$

$$\frac{\partial u_\alpha^*}{\partial t} + u_\beta^* \frac{\partial u_\alpha^*}{\partial x_\beta} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x_\alpha} + \nu \frac{\partial^2 u_\alpha^*}{\partial x_\beta \partial x_\beta}. \tag{2}$$

Decomposition and averaging (Reynolds 1895):

$$u_\alpha^* = \overline{u_\alpha^*} + u_\alpha = U_\alpha + u_\alpha, \quad p^* = \overline{p^*} + p = P + p. \tag{3}$$

$$\overline{p^*}(x_\alpha, t_0) = \lim_{T\to\infty} \frac{1}{2T}\int_{t_0-T}^{t_0+T} p^*(x_\alpha, t)\,dt,$$

$$\overline{p^*}(x_1, x_2, z_0, t) = \lim_{Z\to\infty} \frac{1}{2Z}\int_{z_0-Z}^{z_0+Z} p^*(x_1, x_2, x_3, t)\,dx_3.$$

(3) into (1): $$\frac{\partial(U_\alpha + u_\alpha)}{\partial x_\alpha} = 0. \tag{4}$$

$\overline{(4)}$: $$\frac{\partial U_\alpha}{\partial x_\alpha} = 0, \quad \frac{\partial u_\alpha}{\partial x_\alpha} = 0. \tag{5}$$

(3) into (2): $$\frac{\partial(U_\alpha + u_\alpha)}{\partial t} + (U_\beta + u_\beta)\frac{\partial(U_\alpha + u_\alpha)}{\partial x_\beta} = -\frac{1}{\rho}\frac{\partial(P+p)}{\partial x_\alpha} + \nu\frac{\partial^2(U_\alpha + u_\alpha)}{\partial x_\beta \partial x_\beta}. \tag{6}$$

$\overline{(6)}$: $$U_\beta \frac{\partial U_\alpha}{\partial x_\beta} + \overline{u_\beta \frac{\partial u_\alpha}{\partial x_\beta}} = -\frac{1}{\rho}\frac{\partial P}{\partial x_\alpha} + \nu\frac{\partial^2 U_\alpha}{\partial x_\beta \partial x_\beta}, \tag{7}$$

or $$U_\beta \frac{\partial U_\alpha}{\partial x_\beta} = \frac{\partial T_{\alpha\beta}}{\partial x_\beta}; \tag{8}$$

$$T_{\alpha\beta} = -\delta_{\alpha\beta}\frac{P}{\rho} + \nu S_{\alpha\beta} - \overline{u_\alpha u_\beta}, \quad S_{\alpha\beta} = \frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha}. \tag{9}$$

(6) − (7): $$\frac{\partial u_\alpha}{\partial t} + U_\beta \frac{\partial u_\alpha}{\partial x_\beta} + u_\beta \frac{\partial U_\alpha}{\partial x_\beta} + u_\beta\frac{\partial u_\alpha}{\partial x_\beta} - \overline{u_\beta \frac{\partial u_\alpha}{\partial x_\beta}} = -\frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \nu\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta}. \tag{10}$$

or $$\frac{\partial u_\alpha}{\partial t} + U_\beta\frac{\partial u_\alpha}{\partial x_\beta} = \frac{\partial}{\partial x_\beta}\left\{-\delta_{\alpha\beta}\frac{p}{\rho} + \nu s_{\alpha\beta} - U_\alpha u_\beta - u_\alpha u_\beta + \overline{u_\alpha u_\beta}\right\}, \tag{11}$$

$$s_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}. \tag{12}$$

$\overline{u_\gamma \cdot (10)} + \overline{u_\alpha \cdot (10)\, \alpha \to \gamma}$:

$$U_\beta \frac{\partial \overline{u_\gamma u_\alpha}}{\partial x_\beta} + \left\{\overline{u_\alpha u_\beta}\frac{\partial U_\gamma}{\partial x_\beta} + \overline{u_\beta u_\gamma}\frac{\partial U_\alpha}{\partial x_\beta}\right\} + \frac{\partial \overline{u_\alpha u_\beta u_\gamma}}{\partial x_\beta} = -\frac{1}{\rho}\left\{\overline{u_\alpha \frac{\partial p}{\partial x_\gamma}} + \overline{u_\gamma \frac{\partial p}{\partial x_\alpha}}\right\} + \nu\left\{\overline{u_\alpha \frac{\partial^2 u_\gamma}{\partial x_\beta \partial x_\beta}} + \overline{u_\gamma \frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta}}\right\}. \tag{13}$$

Kinetic energy equations

$$U_\alpha \cdot (8): \quad U_\beta \frac{\partial}{\partial X_\beta}\left(\frac{1}{2} U_\alpha U_\alpha\right) = \frac{\partial}{\partial X_\beta}\left(T_{\alpha\beta} U_\alpha\right) - T_{\alpha\beta}\frac{\partial U_\alpha}{\partial X_\beta} ;$$

$$T_{\alpha\beta}\frac{\partial U_\alpha}{\partial X_\beta} = -\delta_{\alpha\beta}\frac{P}{\rho}\frac{\partial U_\alpha}{\partial X_\beta} + \nu\left(\frac{\partial U_\alpha}{\partial X_\beta} + \frac{\partial U_\beta}{\partial X_\alpha}\right)\frac{\partial U_\alpha}{\partial X_\beta} - \overline{u_\alpha u_\beta}\frac{\partial U_\alpha}{\partial X_\beta}$$

$$= -\frac{P}{\rho}\frac{\partial U_\alpha}{\partial X_\alpha} + \frac{1}{2}\nu\left(\frac{\partial U_\alpha}{\partial X_\beta} + \frac{\partial U_\beta}{\partial X_\alpha}\right)\left(\frac{\partial U_\alpha}{\partial X_\beta} + \frac{\partial U_\beta}{\partial X_\alpha}\right) - \frac{1}{2}\overline{u_\alpha u_\beta}\left(\frac{\partial U_\alpha}{\partial X_\beta} + \frac{\partial U_\beta}{\partial X_\alpha}\right)$$

$$= \frac{1}{2}\nu S_{\alpha\beta} S_{\alpha\beta} - \frac{1}{2}\overline{u_\alpha u_\beta} S_{\alpha\beta},$$

$$\frac{1}{2} S_{\alpha\beta} S_{\alpha\beta} = \text{dissipation function} = 2\left(\frac{\partial U_1}{\partial X_1}\right)^2 + 2\left(\frac{\partial U_2}{\partial X_2}\right)^2 + 2\left(\frac{\partial U_3}{\partial X_3}\right)^2$$
$$+\left(\frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_2}\right)^2 + \left(\frac{\partial U_3}{\partial X_1} + \frac{\partial U_1}{\partial X_3}\right)^2 + \left(\frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1}\right)^2. \qquad (14)$$

$$U_\beta \frac{\partial}{\partial X_\beta}\left(\frac{1}{2} U_\alpha U_\alpha\right) = \frac{\partial}{\partial X_\beta}\left\{-U_\beta \frac{P}{\rho} + U_\alpha\left(\nu S_{\alpha\beta} - \overline{u_\alpha u_\beta}\right)\right\} - \frac{1}{2} S_{\alpha\beta}\left(\nu S_{\alpha\beta} - \underline{\overline{u_\alpha u_\beta}}\right). \quad (15)$$

Put $\gamma = \alpha$ in (13) and $u_\alpha u_\alpha = q^2$:

$$U_\beta \frac{\partial}{\partial X_\beta}\left(\frac{1}{2}\overline{q^2}\right) = -\overline{u_\alpha u_\beta}\frac{\partial U_\alpha}{\partial X_\beta} - \frac{\partial}{\partial X_\beta}\left(\frac{1}{2}\overline{u_\beta q^2}\right) - \frac{1}{\rho}\frac{\partial \overline{u_\alpha p}}{\partial X_\alpha} + \nu \overline{u_\alpha \frac{\partial^2 u_\alpha}{\partial X_\beta \partial X_\beta}},$$

$$u_\alpha \frac{\partial^2 u_\alpha}{\partial X_\beta \partial X_\beta} = u_\alpha \frac{\partial}{\partial X_\beta}\left(\frac{\partial u_\alpha}{\partial X_\beta} + \frac{\partial u_\beta}{\partial X_\alpha}\right)$$

$$= \frac{\partial}{\partial X_\beta}\left\{u_\alpha\left(\frac{\partial u_\alpha}{\partial X_\beta} + \frac{\partial u_\beta}{\partial X_\alpha}\right)\right\} - \frac{\partial u_\alpha}{\partial X_\beta}\left(\frac{\partial u_\alpha}{\partial X_\beta} + \frac{\partial u_\beta}{\partial X_\alpha}\right)$$

$$= \frac{\partial}{\partial X_\beta}\left(u_\alpha s_{\alpha\beta}\right) - \frac{1}{2} s_{\alpha\beta} s_{\alpha\beta}.$$

$$U_\beta \frac{\partial}{\partial X_\beta}\left(\frac{1}{2}\overline{q^2}\right) = \underline{-\frac{1}{2}\overline{u_\alpha u_\beta} S_{\alpha\beta}} - \frac{\partial}{\partial X_\beta}\left\{\overline{u_\beta\left(\frac{p}{\rho} + \frac{q^2}{2}\right)} - \nu \overline{u_\alpha s_{\alpha\beta}}\right\} - \frac{1}{2}\nu \overline{s_{\alpha\beta} s_{\alpha\beta}}. \quad (16)$$

(15) + (16):

$$U_\beta \frac{\partial}{\partial X_\beta}\frac{1}{2}\left(U_\alpha U_\alpha + \overline{q^2}\right) = -\frac{\partial}{\partial X_\beta}\left\{U_\beta \frac{P}{\rho} + \overline{u_\beta\left(\frac{p}{\rho} + \frac{q^2}{2}\right)} + U_\alpha \overline{u_\alpha u_\beta} - \nu\left(U_\alpha S_{\alpha\beta} + \overline{u_\alpha s_{\alpha\beta}}\right)\right\}$$
$$-\frac{1}{2}\nu\left(S_{\alpha\beta} S_{\alpha\beta} + \overline{s_{\alpha\beta} s_{\alpha\beta}}\right). \qquad (17)$$

Pressure equations

$$\operatorname{div}(6): \quad \frac{\partial^2\{(U_\alpha + u_\alpha)(U_\beta + u_\beta)\}}{\partial X_\alpha \partial X_\beta} = -\frac{1}{\rho}\frac{\partial^2 (P+p)}{\partial X_\alpha \partial X_\alpha},$$

$$\frac{1}{\rho}(P+p) = -\frac{1}{4\pi}\int \frac{\partial^2\{(U_\alpha + u_\alpha)(U_\beta + u_\beta)\}}{\partial X'_\alpha \partial X'_\beta}\frac{d\tau(\vec{X}')}{|\vec{X}' - \vec{X}|}. \qquad (18)$$

Vorticity equations

$$\Omega_\alpha = \varepsilon_{\alpha\beta\gamma}\frac{\partial U_\gamma}{\partial x_\beta}, \qquad \omega_\alpha = \varepsilon_{\alpha\beta\gamma}\frac{\partial u_\gamma}{\partial x_\beta}. \tag{19}$$

curl (6) :

$$\frac{\partial(\Omega_\alpha+\omega_\alpha)}{\partial t} + (U_\beta+u_\beta)\frac{\partial(\Omega_\alpha+\omega_\alpha)}{\partial x_\beta} = (\Omega_\beta+\omega_\beta)\frac{\partial(U_\alpha+u_\alpha)}{\partial x_\beta} + \nu\frac{\partial^2(\Omega_\alpha+\omega_\alpha)}{\partial x_\beta\partial x_\beta} \tag{20}$$

$$\begin{aligned} U_\beta\frac{\partial}{\partial x_\beta}\left(\tfrac{1}{2}\Omega_\alpha\Omega_\alpha\right) &= \tfrac{1}{2}\Omega_\alpha\Omega_\beta S_{\alpha\beta} - \frac{\partial}{\partial x_\beta}\left(\Omega_\alpha\overline{\omega_\alpha u_\beta}\right) + \overline{\omega_\alpha u_\beta}\frac{\partial\Omega_\alpha}{\partial x_\beta} \\ &\quad + \tfrac{1}{2}\Omega_\alpha\overline{\omega_\beta s_{\alpha\beta}} + \nu\frac{\partial^2}{\partial x_\beta\partial x_\beta}\left(\tfrac{1}{2}\Omega_\alpha\Omega_\alpha\right) - \nu\frac{\partial\Omega_\alpha}{\partial x_\beta}\frac{\partial\Omega_\alpha}{\partial x_\beta}, \end{aligned} \tag{21}$$

$$\begin{aligned} U_\beta\frac{\partial}{\partial x_\beta}\left(\tfrac{1}{2}\overline{\omega_\alpha\omega_\alpha}\right) &= -\overline{\omega_\alpha u_\beta}\frac{\partial\Omega_\alpha}{\partial x_\beta} - \frac{1}{2}\frac{\partial\overline{\omega_\alpha\omega_\beta u_\beta}}{\partial x_\beta} + \frac{1}{2}\overline{\omega_\alpha\omega_\beta s_{\alpha\beta}} \\ &\quad + \frac{1}{2}\overline{\omega_\alpha\omega_\beta}S_{\alpha\beta} + \frac{1}{2}\Omega_\beta\overline{\omega_\alpha s_{\alpha\beta}} + \nu\frac{\partial^2}{\partial x_\beta\partial x_\beta}\left(\tfrac{1}{2}\overline{\omega_\alpha\omega_\alpha}\right) - \nu\overline{\frac{\partial\omega_\alpha}{\partial x_\beta}\frac{\partial\omega_\alpha}{\partial x_\beta}}. \end{aligned} \tag{22}$$

Thin shear layer approximation

$$V \ll U, \quad \frac{\partial U}{\partial x} \ll \frac{\partial U}{\partial y}, \quad \frac{\partial\overline{v^2}}{\partial x} \ll \frac{\partial\overline{v^2}}{\partial y}.$$

(5) $$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \underline{\frac{\partial W}{\partial z}} = 0,$$

(7) $$\begin{aligned} U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \underline{W\frac{\partial U}{\partial z}} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \underline{\frac{\partial^2 U}{\partial z^2}}\right) - \frac{\partial\overline{u^2}}{\partial x}^{*} - \frac{\partial\overline{uv}}{\partial y} - \underline{\frac{\partial\overline{uw}}{\partial z}}, \\ U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \underline{W\frac{\partial V}{\partial z}} &= -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \underline{\frac{\partial^2 V}{\partial z^2}}\right) - \frac{\partial\overline{uv}}{\partial x} - \frac{\partial\overline{v^2}}{\partial y} - \underline{\frac{\partial\overline{vw}}{\partial z}}, \\ \underline{U\frac{\partial W}{\partial x}} + \underline{V\frac{\partial W}{\partial y}} + \underline{W\frac{\partial W}{\partial z}} &= \underline{-\frac{1}{\rho}\frac{\partial P}{\partial z}} + \nu\left(\underline{\frac{\partial^2 W}{\partial x^2}} + \underline{\frac{\partial^2 W}{\partial y^2}} + \underline{\frac{\partial^2 W}{\partial z^2}}\right) - \underline{\frac{\partial\overline{uw}}{\partial x}} - \underline{\frac{\partial\overline{vw}}{\partial y}} - \underline{\frac{\partial\overline{w^2}}{\partial z}}. \end{aligned}$$

～～～ deleted by thin shear layer approximation.

——— deleted by 'two-dimensional in mean.

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial\overline{u^2}}{\partial x} - \frac{\partial\overline{uv}}{\partial y},$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} - \frac{\partial\overline{v^2}}{\partial y}. \qquad \frac{P}{\rho} + \overline{v^2} = \frac{P_0}{\rho}.$$

Approximate equations for mean momentum & mean energy :

$$\left(U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)U = -\frac{1}{\rho}\frac{dP_0}{dx} + \frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y} - \overline{uv}\right) - \frac{\partial}{\partial x}\left(\overline{u^2} - \overline{v^2}\right); \tag{23}$$

$$\begin{aligned} \left(U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)\frac{U^2}{2} &= -\frac{U}{\rho}\frac{dP_0}{dx} - \frac{\partial}{\partial y}\left(U\overline{uv}\right) - \frac{\partial}{\partial x}\left\{U\left(\overline{u^2} - \overline{v^2}\right)\right\} \\ &\quad + \overline{uv}\frac{\partial U}{\partial y} + \left(\overline{u^2} - \overline{v^2}\right)\frac{\partial U}{\partial x} + \nu\frac{\partial^2}{\partial y^2}\left(\frac{U^2}{2}\right) - \nu\left(\frac{\partial U}{\partial y}\right)^2. \end{aligned} \tag{24}$$

Approximate equation for turbulent energy :

$$\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\frac{\overline{q^2}}{2}=-\overline{uv}\frac{\partial U}{\partial y}-\left(\overline{u^2}-\overline{v^2}\right)\frac{\partial U}{\partial x}-\frac{1}{2}\nu\overline{s_{\alpha\beta}s_{\alpha\beta}}$$

$$-\frac{\partial}{\partial y}\left\{\overline{v\left(\frac{p}{\rho}+\frac{q^2}{2}\right)}-\nu\frac{\partial}{\partial y}\left(\overline{v^2}+\frac{\overline{q^2}}{2}\right)\right\} \qquad (25)$$

$$-\overline{uv}\frac{\partial U}{\partial y}\sim\frac{u_0^2U_0}{\ell},\quad \frac{1}{2}\nu\overline{s_{\alpha\beta}s_{\alpha\beta}}\sim\frac{\nu u_0^2}{\lambda^2};$$

$$\frac{u_0^2U_0}{\ell}\sim\frac{\nu u_0^2}{\lambda^2}:\quad \frac{\lambda}{\ell}\sim\left(\frac{U_0\ell}{\nu}\right)^{-\frac{1}{2}};$$

$$\overline{v\left(\frac{p}{\rho}+\frac{q^2}{2}\right)}\sim u_0^3,\quad \nu\frac{\partial}{\partial y}\left(\overline{v^2}+\frac{\overline{q^2}}{2}\right)\sim\frac{\nu u_0^2}{\ell}:$$

$$\nu\frac{\partial}{\partial y}\left(\overline{v^2}+\frac{\overline{q^2}}{2}\right)\Big/\overline{v\left(\frac{p}{\rho}+\frac{q^2}{2}\right)}\sim\left(\frac{u_0\ell}{\nu}\right)^{-1}!$$

Approximate equations for mean & turbulent vorticity :

$$\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\frac{\Omega_3^2}{2}=-\frac{\partial}{\partial y}\left(\Omega_3\overline{v\omega_3}\right)+\overline{v\omega_3}\frac{\partial\Omega_3}{\partial y}-\frac{\Omega_3}{2}\overline{\omega_\beta s_{3\beta}}, \qquad (26)$$

$$\frac{1}{2}\overline{\omega_\alpha\omega_\beta s_{\alpha\beta}}=\nu\overline{\frac{\partial\omega_\alpha}{\partial x_\beta}\frac{\partial\omega_\alpha}{\partial x_\beta}}. \qquad (27)$$

Channel flows

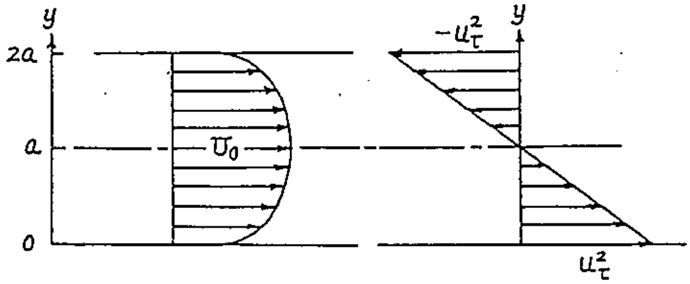
$$\bar{U}_\alpha = \bar{U}(y), 0, 0: \quad \bar{U}_\beta \frac{\partial \bar{U}_\alpha}{\partial x_\beta} = 0 \quad \text{or} \quad \bar{U}\frac{\partial \bar{U}}{\partial x} + \bar{V}\frac{\partial \bar{U}}{\partial y} = 0;$$

$$-\frac{1}{\rho}\frac{dP_0}{dx} + \frac{d}{dy}\left(\nu \frac{d\bar{U}}{dy} - \overline{uv}\right) = 0,$$

$$-\frac{y}{\rho}\frac{dP_0}{dx} + \nu\frac{d\bar{U}}{dy} - \overline{uv} = \frac{\tau_w}{\rho}.$$

$$y = a: \; -\frac{a}{\rho}\frac{dP_0}{dx} = \frac{\tau_w}{\rho} = u_\tau^2 \; ; \quad u_\tau = \text{friction velocity}.$$

$$\nu\frac{d\bar{U}}{dy} - \overline{uv} = u_\tau^2\left(1 - \frac{y}{a}\right).$$



$$R = \frac{u_\tau a}{\nu}, \quad y^+ = \frac{u_\tau y}{\nu}: \quad \frac{d}{dy^+}\frac{\bar{U}}{u_\tau} - \frac{\overline{uv}}{u_\tau^2} = 1 - \frac{1}{R}y^+.$$

$$R \longrightarrow \infty: \quad \frac{d}{dy^+}\frac{\bar{U}}{u_\tau} - \frac{\overline{uv}}{u_\tau^2} = 1 \quad (\text{inner solution}).$$

$$\text{smooth wall}: \quad \frac{\bar{U}}{u_\tau} = f(y^+), \quad -\frac{\overline{uv}}{u_\tau^2} = 1 - \frac{df}{dy^+};$$

$$\frac{d\bar{U}}{dy} = \frac{u_\tau^2}{\nu}\frac{df}{dy^+}: \quad \frac{df}{dy^+} = 1, \quad f = y^+ \text{ for } y^+ \to 0.$$

$$Y = \frac{y}{a}: \quad \frac{1}{R}\frac{d}{dY}\frac{\bar{U}}{u_\tau} - \frac{\overline{uv}}{u_\tau^2} = 1 - Y.$$

$$R \longrightarrow \infty: \quad -\frac{\overline{uv}}{u_\tau^2} = 1 - Y \quad (\text{outer solution}).$$

Experimental results suggest $\dfrac{\bar{U} - \bar{U}_0}{u_\tau} = F(Y)$;

Energy equation suggests $\dfrac{d\bar{U}}{dy} = \dfrac{u_\tau}{a}\dfrac{dF}{dY}$, $\dfrac{\bar{U} - \bar{U}_0}{u_\tau} = F(Y)$.

Matching $d\bar{U}/dy$ in the region of overlap:

$$\frac{u_\tau^2}{\nu}\frac{df}{dy^+} = \frac{u_\tau}{a}\frac{dF}{dY}, \qquad y^+\frac{df}{dy^+} = Y\frac{dF}{dY} = \frac{1}{k};$$

$$f(y^+) = \frac{1}{k}\log y^+ + C \quad \text{for } y^+ \gg 1,$$

$$F(Y) = \frac{1}{k}\log Y + D \quad \text{for } Y \ll 1.$$

Wall law :

$$\frac{\bar{U}}{u_\tau} = f(y^+): \quad f = y^+, \ y^+ \to 0 \ \text{(viscous sublayer)},$$

$$f = \frac{1}{k}\log y^+ + C, \ y^+ \gg 1 \ \text{(inertial sublayer)}.$$

velocity-defect law :

$$\frac{\bar{U} - \bar{U}_0}{u_\tau} = F(Y): \quad F = \frac{1}{k}\log Y + D, \ \ Y \ll 1 \ \text{(inertial s.-l.)}.$$

subtraction leads to friction law :

$$\frac{\bar{U}_0}{u_\tau} = \frac{1}{k}\log R + C - D, \ \text{or} \ \frac{\bar{U}_0}{u_\tau} = \sqrt{\frac{2}{c_f}} = \frac{1}{k}\log\frac{\bar{U}_0 a}{\nu}\sqrt{\frac{c_f}{2}} + C - D.$$

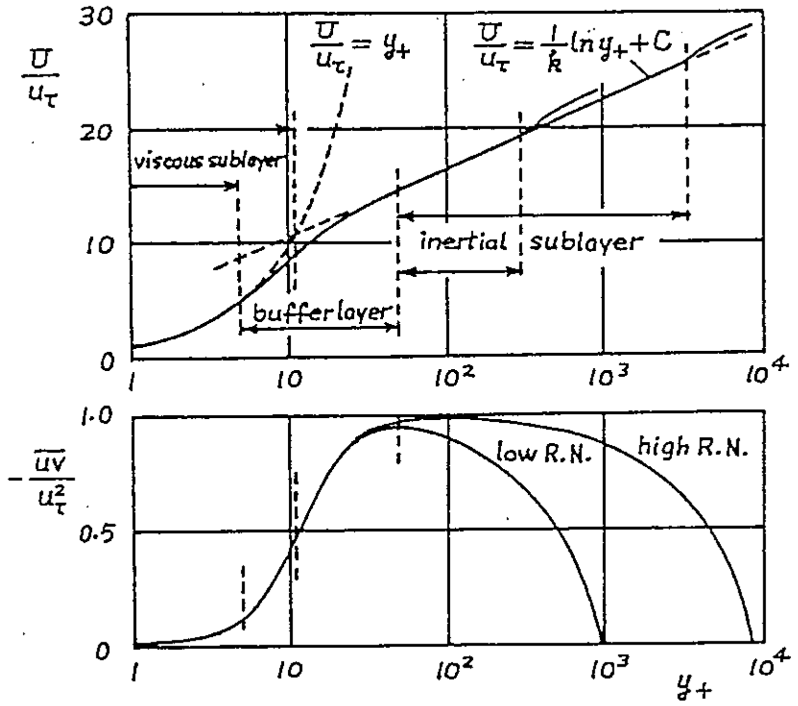
Constants :

$k = 0.41, \quad C = 5.0, \quad D = 0.0$ for channel flow.
(two-dimensional or axisymmetric)

Reynolds stress :

$$-\frac{\overline{uv}}{u_\tau^2} = \alpha y^{+3} \ (\alpha = 0.0011) \ \text{for} \ y^+ \to 0,$$

$$-\frac{\overline{uv}}{u_\tau^2} = 1 \ \text{both for} \ y^+ \gg 1 \ \text{and} \ Y \ll 1.$$

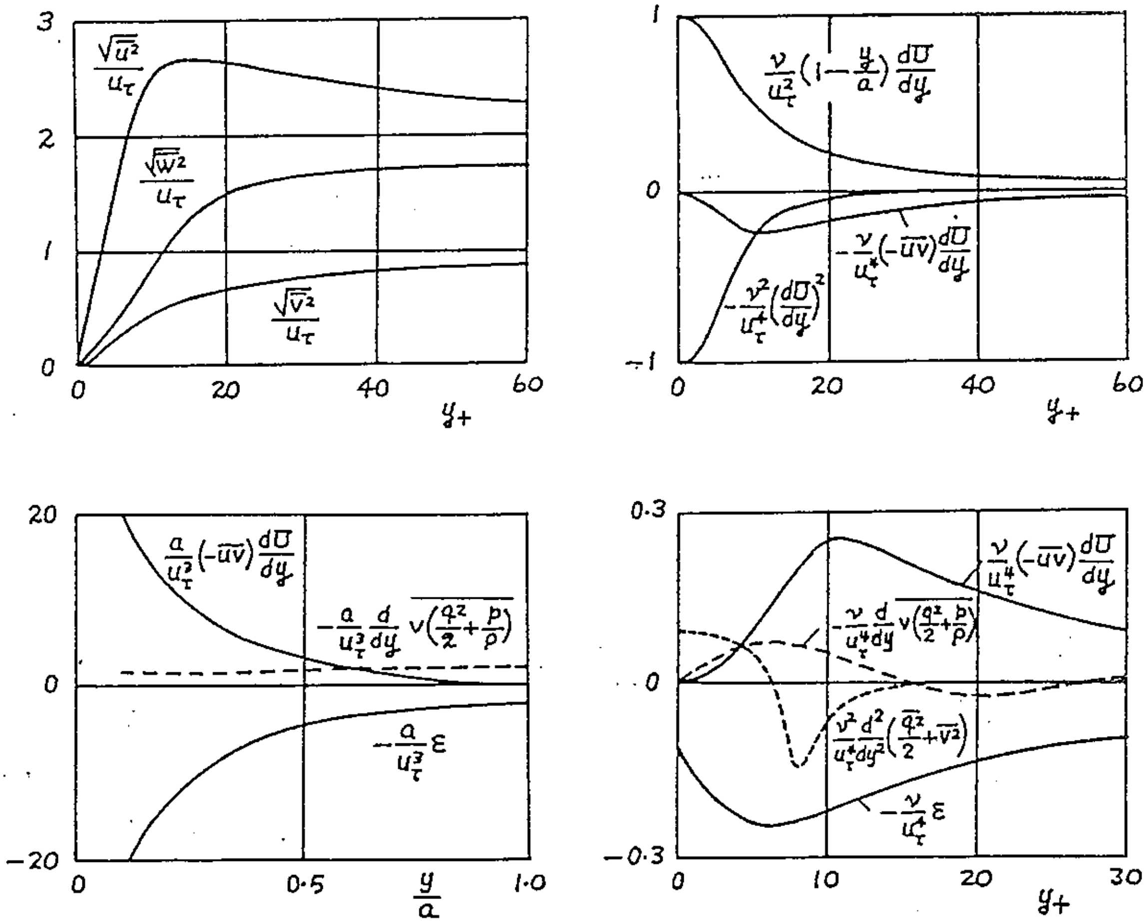


Energy equations:

$$u_\tau^2\left(1-\frac{y}{a}\right)\frac{d\overline{U}}{dy} - \nu\left(\frac{d\overline{U}}{dy}\right)^2 - \left(-\overline{uv}\right)\frac{d\overline{U}}{dy} = 0,$$

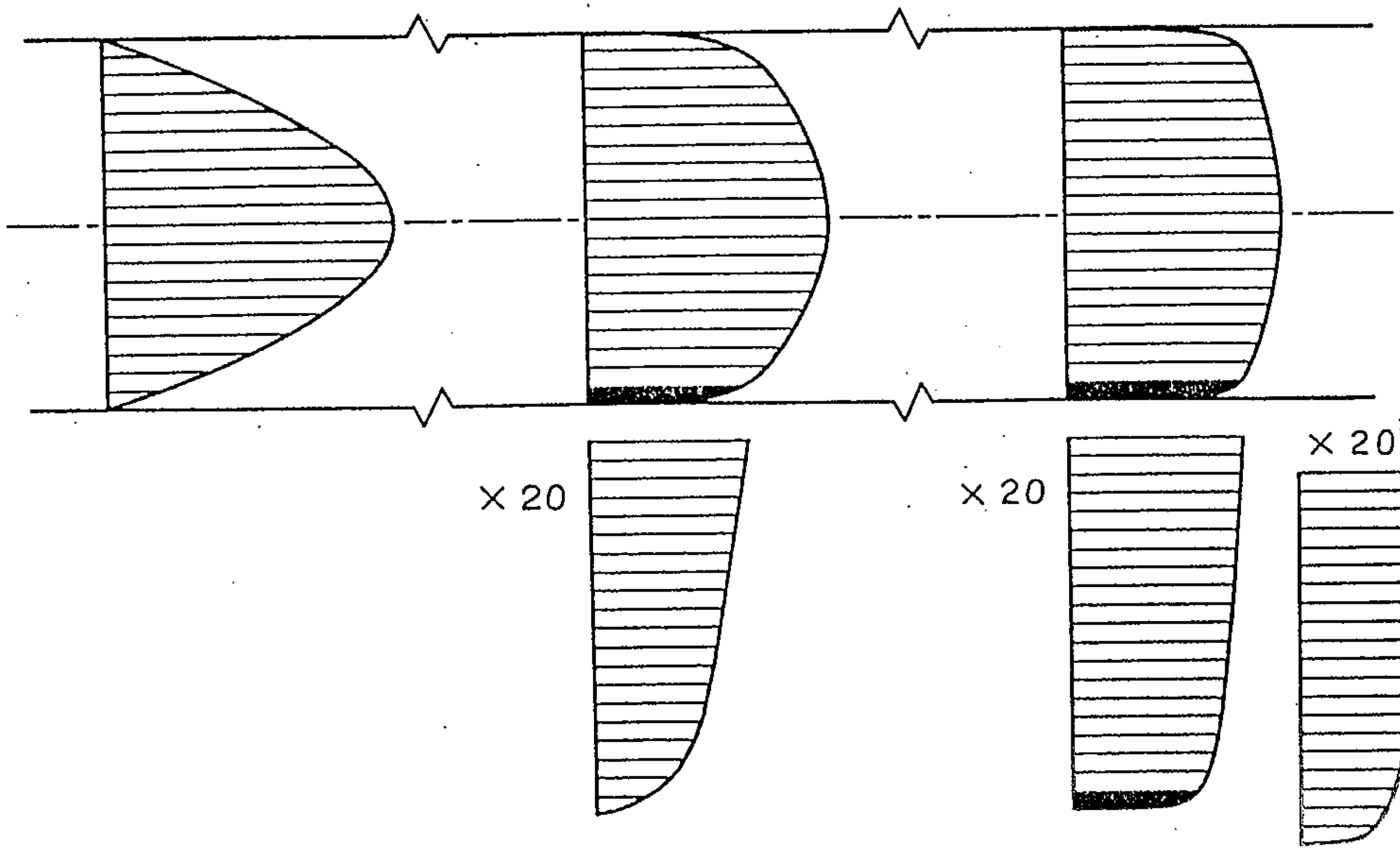
$$-\overline{uv}\,\frac{d\overline{U}}{dy} - \frac{d}{dy}\left\{\overline{v\left(\frac{p}{\rho}+\frac{q^2}{2}\right)} - \nu\frac{d}{dy}\left(\overline{v^2}+\frac{\overline{q^2}}{2}\right)\right\} - \varepsilon = 0,$$

$$\varepsilon = \frac{1}{2}\nu\,\overline{s_{\alpha\beta}s_{\alpha\beta}}.$$



$$\text{Reynolds number} = \frac{\text{representative length} \times \text{representative velocity}}{\text{kinematic viscosity}}$$

As Reynolds number increases ⟶



Effect of wall roughness

$$\frac{\tau_w}{\rho} = u_\tau^2 = \text{viscous shear stress} + \text{component of pressure force.}$$

$$\frac{U}{u_\tau} = f(y^+, h^+), \quad y^+ = \frac{u_\tau y}{\nu}, \quad h^+ = \frac{u_\tau h}{\nu};$$

$$\frac{U - U_0}{u_\tau} = F(Y), \quad Y = \frac{y}{a}.$$

- In the inertial subrange, experiments indicate (Clauser 1954)

$$f(y^+, h^+) = \frac{1}{k}\log y^+ + C - K(h^+),$$

where

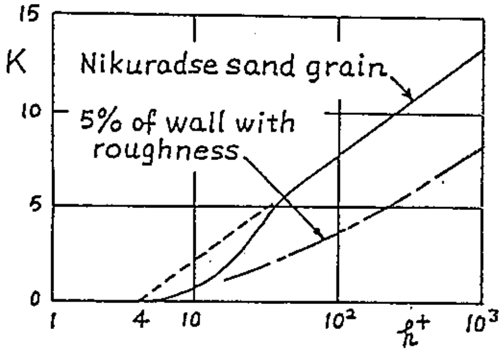
$$K(h^+) = \frac{1}{k}\log h^+ - E$$

for fully rough regime ($h^+ > 70$).

Velocity profile then takes a form

$$\frac{U}{u_\tau} = \frac{1}{k}\log\frac{y}{h} + C + E,$$

which is independent of viscosity.



An alternative form with the roughness length $z_0 = h\,e^{-k(C+E)}$

$$\frac{U}{u_\tau} = \frac{1}{k}\log\frac{y}{z_0}$$

is favored by meteorologists. For Nikuradse sand-grain roughness $E = 3{\cdot}38$, $z_0 = 0{\cdot}032\,h$.

- For a certain kind of roughness typified by closely spaced grooves K is independent of h^+ but dependent on au_τ/ν, defining a 'd-type' roughness, in contrast to a 'k-type' roughness such as Nikuradse sand-grain roughness (Perry et al. 1969).
- The origin of y-coordinate is determined in such a way that the straight-line logarithmic portion of velocity profile is obtained. This shift of origin, if properly determined, amounts to about $0{\cdot}7h$ above the valley of the roughness element. Physically, the height may be regarded as the elevation where the mean drag appears to act on the flow well above the roughness elements.

Heat transfer in turbulent channel flows

Prandtl number $\sigma = \dfrac{\nu}{\kappa} = \dfrac{\text{kinematic viscosity}}{\text{thermometric conductivity}}$.

$\rho c_p \kappa \left(-\dfrac{d\Theta}{dy}\right)_w = \rho c_p u_\tau \theta_\tau$, θ_τ = friction temperature.

wall law $\dfrac{\Theta_w - \Theta}{\theta_\tau} = g(y^+, \sigma)$, defect law $\dfrac{\Theta_0 - \Theta}{\theta_\tau} = G(Y)$.

$g(y^+, \sigma) = \dfrac{1}{k_\theta}\log y^+ + C_\theta \;\; (y^+ \gg 1)$, $G(Y) = \dfrac{1}{k_\theta}\log Y + D_\theta \;\; (Y \ll 1)$;

$$\frac{k}{k_\theta} = 0.85, \quad C_\theta = C_\theta(\sigma), \quad D_\theta = D.$$

$\dfrac{\Theta_w - \Theta_0}{\theta_\tau} = \dfrac{1}{k_\theta}\log\dfrac{u_\tau a}{\nu} + C_\theta + D$, or $c_h = \dfrac{u_\tau \theta_\tau}{U_0(\Theta_w - \Theta_0)} = \sqrt{\dfrac{c_f}{2}} \Big/ \left[\dfrac{1}{k_\theta}\log\dfrac{U_0 a}{\nu}\sqrt{\dfrac{c_f}{2}} + C_\theta + \right.$

$\sigma \gg 1$: $\nu_T = \nu\alpha y^{+3}$, $\kappa_T = \nu\alpha_\theta y^{+3}$, $\dfrac{\alpha}{\alpha_\theta} = \dfrac{k}{k_\theta} = 0.85$ in viscous sublayer.

Within thermal diffusion sublayer $0 \leqslant y^+ \leqslant y_1^+$:

$$-\kappa\frac{d\Theta}{dy} = u_\tau\theta_\tau, \quad \frac{dg}{dy^+} = \sigma, \quad g(y^+) = \sigma y^+.$$

$$\kappa = \nu\alpha_\theta y^{+3} \text{ at } y^+ = y_1^+ : \quad y_1^+ = \alpha_\theta^{-\frac{1}{3}}\sigma^{-\frac{1}{3}}.$$

Outside thermal diffusion sublayer but within viscous sublayer

$$y_1^+ \leqslant y^+ \ll y_2^+ :$$

$$-\kappa_T\frac{d\Theta}{dy} = u_\tau\theta_\tau, \quad \frac{dg}{dy^+} = \frac{1}{\alpha_\theta y^{+3}}, \quad g(y^+) = -\frac{1}{2\alpha_\theta y^{+2}} + H.$$

matching at $y^+ = y_1^+$: $\quad H = \dfrac{1}{2\alpha_\theta y_1^{+2}} + \sigma y_1^+ = \dfrac{3}{2}\alpha_\theta^{-\frac{1}{3}}\sigma^{\frac{2}{3}}.$

$\nu\alpha_\theta y^{+3} = \nu k_\theta y^+$ at $y^+ = y_2^+$: $\quad y_2^+ = k_\theta^{\frac{1}{2}}\alpha_\theta^{-\frac{1}{2}} = k^{\frac{1}{2}}\alpha^{-\frac{1}{2}}.$

matching at $y^+ = y_2^+$: $\quad -\dfrac{1}{2\alpha_\theta y_2^{+2}} + \dfrac{3}{2}\alpha_\theta^{-\frac{1}{3}}\sigma^{\frac{2}{3}} = \dfrac{1}{k_\theta}\log y_2^+ + C_\theta;$

$$C_\theta = \frac{3}{2}\alpha_\theta^{-\frac{1}{3}}\sigma^{\frac{2}{3}} - \frac{1}{2k_\theta}\left(1 + \log\frac{k_\theta}{\alpha_\theta}\right) = 13.8\,\sigma^{\frac{2}{3}} - 7.2 \longrightarrow c_1\sigma^{\frac{2}{3}} - c_2.$$

$\sigma \ll 1$: thermal diffusion sublayer is thicker than viscous sublayer.

$$\kappa = \nu k_\theta y^+ \text{ at } y^+ = y_3^+ : \qquad y_3^+ = \frac{1}{k_\theta\sigma}.$$

matching at $y^+ = y_3^+$: $\quad \sigma y_3^+ = \dfrac{1}{k_\theta}\log y_3^+ + C_\theta;$

$$C_\theta = \frac{1}{k_\theta}\log\sigma + \frac{1}{k_\theta}\left(1 + \log k_\theta\right) = 2.1\log\sigma + 0.56 \rightarrow c_3\log\sigma + c_4.$$

Comparison with experiments yields $C_\theta = 12.5\,\sigma^{\frac{2}{3}} + 2.12\log\sigma - E$ for the range $6\times10^{-3} < \sigma < 10^6$, where E is 5.3 for $\sigma \geqslant 0.7$ and 1.5 for $\sigma \ll 1$.

Boundary layer flows

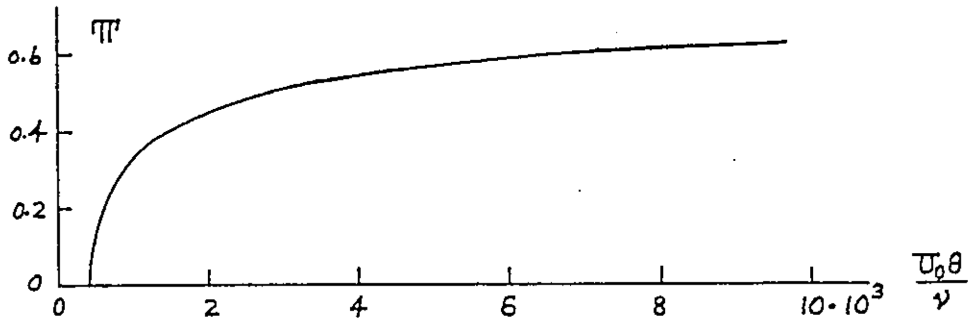
- viscous sublayer: $y \leqslant (0{\cdot}001 - 0{\cdot}01)\,\delta$,
  constant shear stress: $y \leqslant (0{\cdot}1 - 0{\cdot}2)\,\delta$,
  intermittency: $y \geqslant (0{\cdot}1 - 0{\cdot}2)\,\delta$.
- law of wall + law of wake (Coles 1956):

$$\frac{U}{u_\tau} = f\left(\frac{u_\tau y}{\nu}\right) + \frac{2\Pi(x)}{k}\sin^2\frac{\pi y}{2\delta}\,;$$

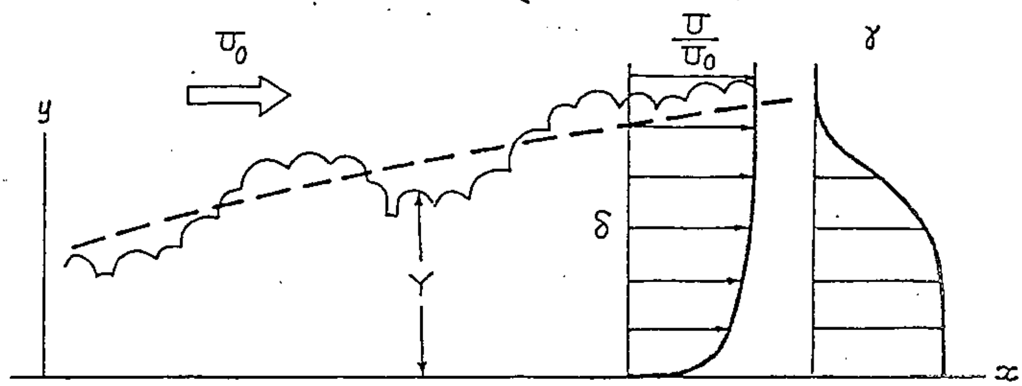
$$\frac{U_0}{u_\tau} = \frac{1}{k}\log\frac{u_\tau\delta}{\nu} + C + 2\frac{\Pi}{k}, \qquad \frac{U_0}{u_\tau}\frac{\delta^*}{\delta} = \frac{1}{k}(1+\Pi),$$

$$\frac{U_0}{u_\tau}\frac{\theta}{\delta} = \frac{U_0}{u_\tau}\frac{\delta^*}{\delta} - \frac{1}{k^2}\frac{u_\tau}{U_0}\left(2 + 3{\cdot}1790\,\Pi + \frac{3}{2}\Pi^2\right).$$

- self-preserving flow: Π = constant (independent of x).
- example: boundary layer in zero pressure gradient.



- intermittency of free boundary (Corrsin 1943, Townsend 1949).

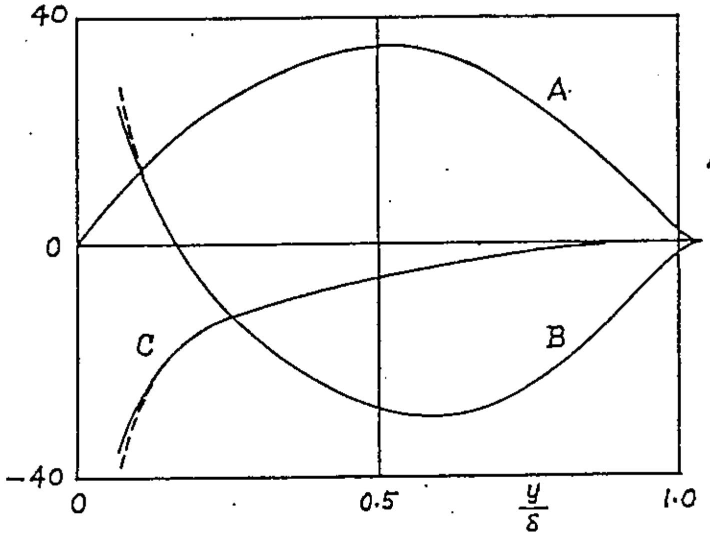


intermittency factor $\gamma(x,y)$ is the probability of finding the interface at a distance greater than y ($Y > y$), or $-\frac{\partial\gamma}{\partial y}$ is the probability density of Y, so that $\overline{Y}(x) = -\int_0^\infty y\,\frac{\partial\gamma}{\partial y}\,dy = \int_0^\infty \gamma(x,y)\,dy$.

- *energy balance :*

$$-\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\frac{U^2}{2}+\frac{\partial}{\partial y}\left\{U\left(-\overline{uv}+\nu\frac{\partial U}{\partial y}\right)\right\}-\left(-\overline{uv}+\nu\frac{\partial U}{\partial y}\right)\frac{\partial U}{\partial y}=0,$$

$$-\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\frac{\overline{q^2}}{2}-\frac{\partial}{\partial y}\overline{v\left(\frac{p}{\rho}+\frac{q^2}{2}\right)}+\left(-\overline{uv}\right)\frac{\partial U}{\partial y}-\frac{1}{2}\nu\overline{s_{\alpha\beta}s_{\alpha\beta}}=0.$$

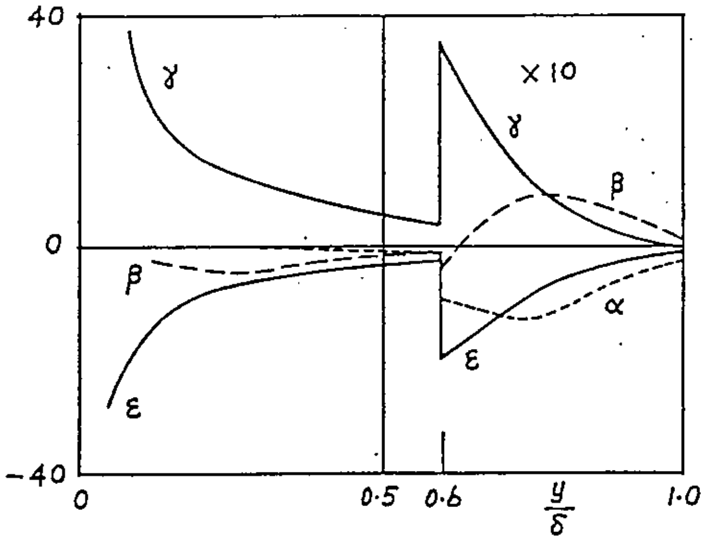


$A+B+C=0:$

$$A=-\frac{\delta}{u_\tau^3}\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\frac{U^2}{2},$$

$$B=+\frac{\delta}{u_\tau^3}\frac{\partial}{\partial y}\left\{U\left(-\overline{uv}\right)\right\},$$

$$C=-\frac{\delta}{u_\tau^3}\left(-\overline{uv}\right)\frac{\partial U}{\partial y}.$$



$\alpha+\beta+\gamma+\varepsilon=0:$

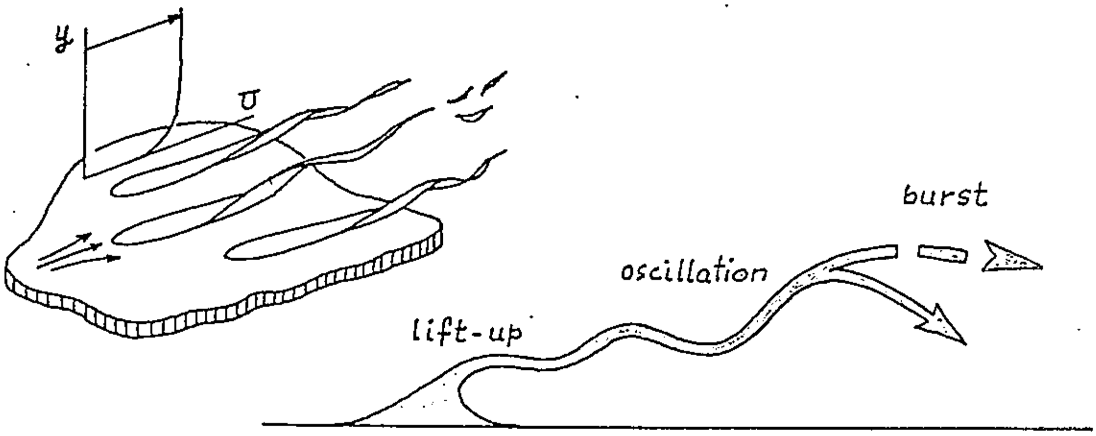
$$\alpha=-\frac{\delta}{u_\tau^3}\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\frac{\overline{q^2}}{2},$$

$$\beta=-\frac{\delta}{u_\tau^3}\frac{\partial}{\partial y}\overline{v\left(\frac{p}{\rho}+\frac{q^2}{2}\right)},$$

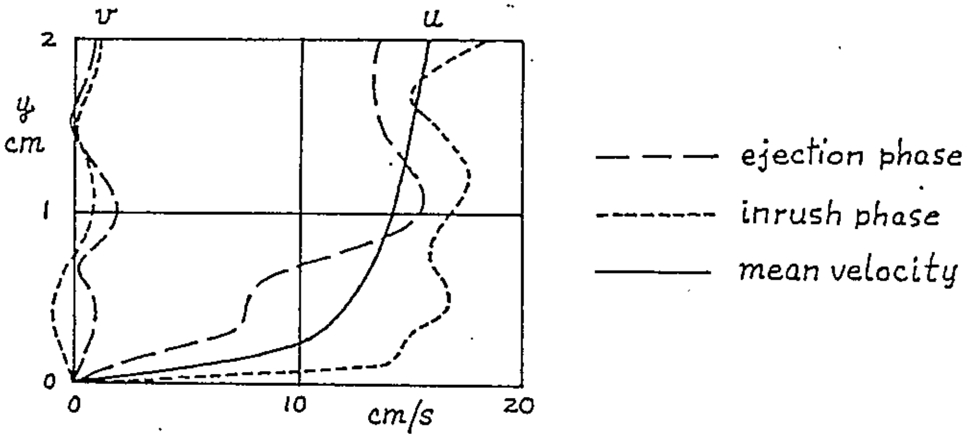
$$\gamma=+\frac{\delta}{u_\tau^3}\left(-\overline{uv}\right)\frac{\partial U}{\partial y},$$

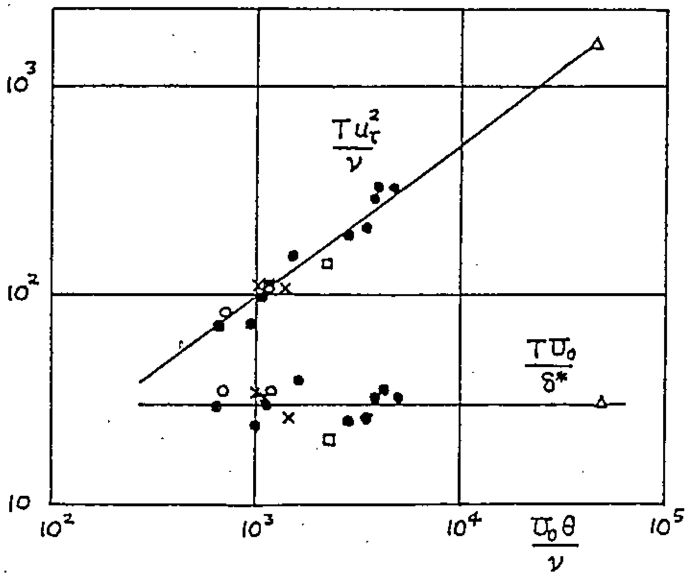
$$\varepsilon=-\frac{\delta}{u_\tau^3}\frac{1}{2}\nu\overline{s_{\alpha\beta}s_{\alpha\beta}}.$$

Viscous sublayer



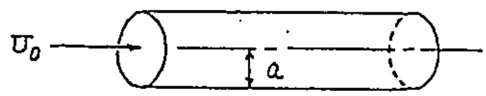
S. J. Kline 1967





$$\frac{\lambda u_\tau}{\nu} \fallingdotseq 100$$

Turbulent flow through a pipe



$a = 1\ \text{cm}, \quad U_0 = 10^3\ \text{cm s}^{-1}.$

$R = 1 \times 10^3 \div 0.14 = 7 \times 10^3.$

$\psi = 0.0665\, R^{-\frac{1}{4}} = 0.0073.$

$u_\tau = \left(\frac{\psi}{2}\right)^{\frac{1}{2}} U_0 = 60.3\ \text{cm s}^{-1},$ smallest eddy size $\frac{4\nu}{u_\tau} = 0.93 \times 10^{-2}\ \text{cm},$

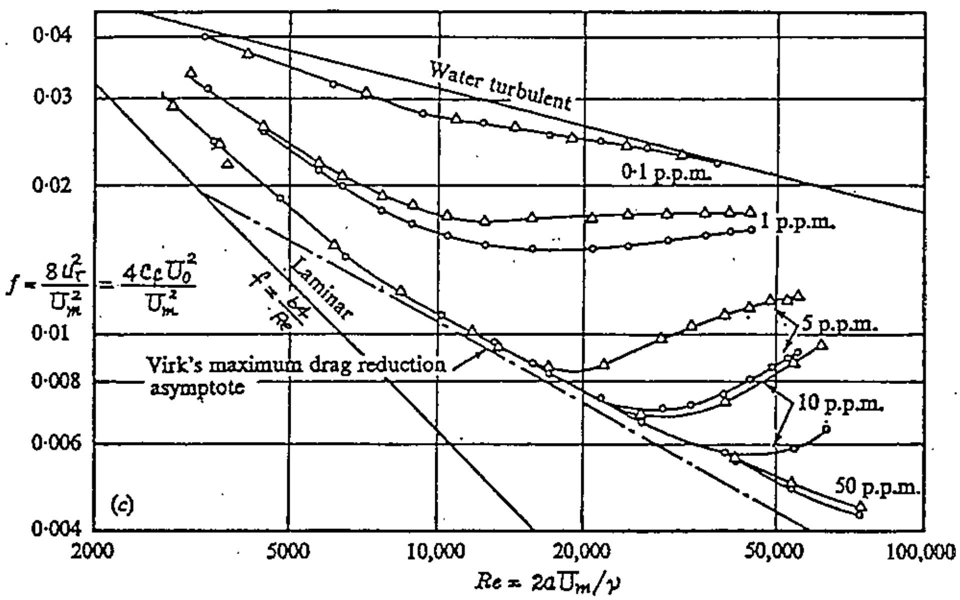
number of eddies $\frac{\pi a^2 \times 10a}{(0.93 \times 10^{-2} a)^3} = 4 \times 10^7,$

number of grid points $4 \times 10^7 \times 4^3 = 2.6 \times 10^9,$

numerical operations $2.6 \times 10^9 \times 10^2 \times 10^2 = 2.6 \times 10^{13},$

computing time $2.6 \times 10^{13} \times 10^{-6}\ \text{s} = 2.6 \times 10^7\ \text{s} = 8$ years.

Drag reduction by polymer additives



Paterson & Abernathy 1970

Integral method of calculating turbulent boundary layer

- Momentum integral relation is obtained by integrating the momentum equation $U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=U_0\frac{dU_0}{dx}+\frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y}-\overline{uv}\right)$ across the boundary layer in the form $\frac{d}{dx}\int_0^\delta U^2dy-U_0\frac{d}{dx}\int_0^\delta U\,dy=U_0\delta\frac{dU_0}{dx}-\frac{\tau_w}{\rho}$, or

$$\boxed{\frac{d\theta}{dx}+(H+2)\frac{\theta}{U_0}\frac{dU_0}{dx}=\frac{u_\tau^2}{U_0^2},}$$

where $U_0\delta^*=\int_0^\infty(U_0-U)\,dy,\quad U_0^2\theta=\int_0^\infty(U_0-U)U\,dy,\quad H=\frac{\delta^*}{\theta},\quad u_\tau^2=\frac{\tau_w}{\rho}.$

- One-parameter family of velocity profiles : $\frac{U}{U_0}=f\left(\frac{y}{\theta},H\right).$

  If $\frac{U}{U_0}=\left(\frac{y}{\delta}\right)^n$: $\frac{\delta^*}{\delta}=\frac{n}{n+1},\ \frac{\theta}{\delta}=\frac{n}{(n+1)(2n+1)},\ H=2n+1,\ \frac{U}{U_0}=\left\{\frac{H}{H(H+1)}\frac{y}{\theta}\right\}^{\frac{H-1}{2}}$

- Wall shear stress : $\frac{u_\tau^2}{U_0^2}=0.123\cdot 10^{-0.678H}R_\theta^{-0.268}\quad\left(R_\theta=\frac{U_0\theta}{\nu}\right).$

  If $n=\frac{1}{7},\ H=1.286,\ \frac{dU_0}{dx}=0$: $\frac{d\theta}{dx}=0.0165\,R_\theta^{-0.268},\quad \theta=0.0474\left(\frac{\nu}{U_0x}\right)^{0.211}x$

- More generally, the profile parameter H is determined from an auxiliary equation

$$\boxed{\theta\frac{dH}{dx}+M\frac{\theta}{U_0}\frac{dU_0}{dx}+N=0,}$$

which is obtained by integrating the momentum equation after multiplication by U or y, or by considering the entrainment at the outer edge. M and N are dependent on H and R_θ.

- When use is made of law of wall plus law of wake $\frac{U}{u_\tau}=\frac{1}{k}\ln\frac{u_\tau y}{\nu}+C+\frac{2B}{k}\sin^2\frac{\pi y}{2\delta}$, the assumptions for velocity profiles and wall shear stress are replaced by

$$\boxed{\frac{\delta^*}{\delta}=\frac{1}{k}\frac{u_\tau}{U_0}(1+B),\quad \frac{\theta}{\delta}=\frac{1}{k}\frac{u_\tau}{U_0}(1+B)-\frac{2}{k^2}\frac{u_\tau^2}{U_0^2}\left(1+1.5895\,B+\frac{3}{4}B^2\right),}$$

and

$$\boxed{\frac{U_0}{u_\tau}=\frac{1}{k}\ln\frac{u_\tau\delta}{\nu}+C+\frac{2B}{k},}$$

so that we have

$$c_{\alpha1}\frac{d\delta}{dx}+c_{\alpha2}\frac{du_\tau}{dx}+c_{\alpha3}\frac{d}{dx}\left(\frac{u_\tau B}{k}\right)=d_\alpha,\quad \alpha=1,2,3.$$

Differential method of calculating turbulent boundary layer

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0,\quad U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{1}{\rho}\frac{dP}{dx}+\frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y}-\overline{uv}\right).$$

$$-\overline{uv}=\nu_T\frac{\partial U}{\partial y}:\quad \nu_T=\text{eddy viscosity (Newtonian constitutive equation)}$$

Mixing length model: $\nu_T=\ell^2\frac{\partial U}{\partial y}$, ℓ = mixing length.

$$\left.\begin{array}{ll} y<\frac{\lambda}{k}\delta: & \ell=ky\left(1-e^{-\frac{y_+}{A}}\right),\\ y>\frac{\lambda}{k}\delta: & \ell=\lambda\delta,\ \lambda=0.085.\end{array}\right\}\quad u_\tau^2=\frac{\tau_w}{\rho},\ y_+=\frac{u_\tau y}{\nu},\ k=0.41,\quad A=\frac{24}{1+20.3\frac{\nu}{\rho u_\tau^3}\frac{dP}{dx}}.$$

One-equation model: $\nu_T=c_1\ell q$, $q^2=\overline{u^2}+\overline{v^2}+\overline{w^2}$

$$U\frac{\partial q^2}{\partial x}+V\frac{\partial q^2}{\partial y}=2\left\{-\overline{uv}\frac{\partial U}{\partial y}-\varepsilon\right\}-\frac{\partial}{\partial y}\overline{v\left(\frac{p}{\rho}+\frac{u^2+v^2+w^2}{2}\right)}\qquad \varepsilon=\frac{\nu}{2}\overline{\left(\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}\right)\left(\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}\right)}$$

$$=2\left\{\nu_T\left(\frac{\partial U}{\partial y}\right)^2-c_2\frac{q^3}{\ell}\right\}+\frac{\partial}{\partial y}\left\{(\nu+c_3\nu_T)\frac{\partial q^2}{\partial y}\right\}.$$

$y_+<2A$: $\ell=ky\left(1-e^{-\frac{y_+}{A}}\right)$, zero-equation model.

$y_+>2A$: ℓ prescribed, $c_1=0.38$, $c_2=c_1^3$, $c_3=0.59$ (Reynolds 1976).

Two-equation model: $\ell\sim\frac{q^3}{\varepsilon}$, $\nu_T\sim\frac{q^4}{\varepsilon}$ or $\ell\sim\frac{q}{\omega}$, $\nu_T\sim\frac{q^2}{\omega}$.

- q^2–ε model: ε = dissipation, $\nu_T=d_1\frac{q^4}{\varepsilon}$;

$$U\frac{\partial q^2}{\partial x}+V\frac{\partial q^2}{\partial y}=2\left\{\nu_T\left(\frac{\partial U}{\partial y}\right)^2-\varepsilon\right\}+\frac{\partial}{\partial y}\left\{(\nu+d_2\nu_T)\frac{\partial q^2}{\partial y}\right\},$$

$$U\frac{\partial\varepsilon}{\partial x}+V\frac{\partial\varepsilon}{\partial y}=\left\{d_3\nu_T\left(\frac{\partial U}{\partial y}\right)^2-d_4\varepsilon\right\}\frac{\varepsilon}{q^2}+\frac{\partial}{\partial y}\left\{(\nu+d_5\nu_T)\frac{\partial\varepsilon}{\partial y}\right\}.$$

$d_1=0.02\,(0.02)$, $d_2=0.9\,(0.8)$, $d_3=2.9\,(2.0)$, $d_4=3.8\,(3.7)$, $d_5=0.9\,(0.8)$

Spalding & Launder 1974 (Reynolds 1976)

- q^2–ω^2 model: $\omega^2=\overline{\omega_x^2}+\overline{\omega_y^2}+\overline{\omega_z^2}$ = mean square vorticity, $\nu_T=\frac{q^2}{2\omega}$;

$$U\frac{\partial q^2}{\partial x}+V\frac{\partial q^2}{\partial y}=\left\{\alpha_1\left|\frac{\partial U}{\partial y}\right|-\beta_1\omega\right\}q^2+\frac{\partial}{\partial y}\left\{(\nu+\sigma_1\nu_T)\frac{\partial q^2}{\partial y}\right\},$$

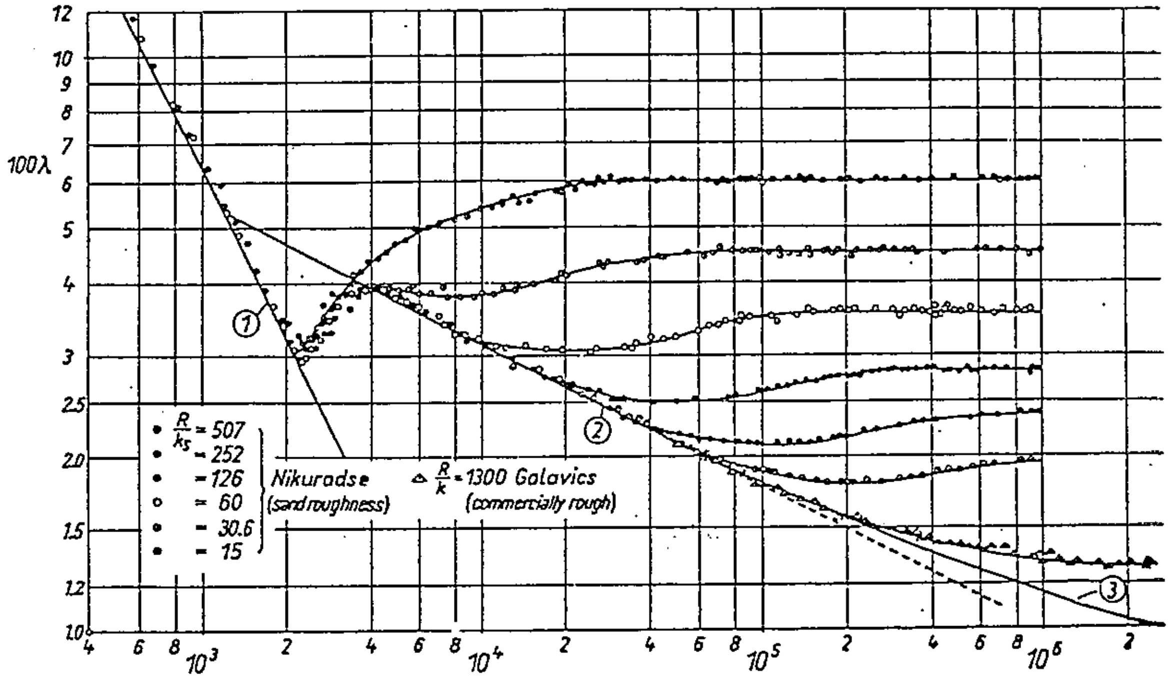
$$U\frac{\partial\omega^2}{\partial x}+V\frac{\partial\omega^2}{\partial y}=\left\{\alpha_2\left|\frac{\partial U}{\partial y}\right|-\beta_2\omega\right\}\omega^2+\frac{\partial}{\partial y}\left\{(\nu+\sigma_2\nu_T)\frac{\partial\omega^2}{\partial y}\right\}.$$

$\alpha_1=0.3$, $\beta_1=0.09$, $\sigma_1=0.5$, $\alpha_2=0.164$, $\beta_2=0.15$, $\sigma_2=0.5$. Saffman & Wilcox 1974

Reynolds stress model: $R_{\alpha\beta}=-\overline{u_\alpha u_\beta}$.

$$U_\beta\frac{\partial R_{\alpha\gamma}}{\partial x_\beta}=\underbrace{-\left(R_{\alpha\beta}\frac{\partial U_\gamma}{\partial x_\beta}+R_{\beta\gamma}\frac{\partial U_\alpha}{\partial x_\beta}\right)}_{\text{production by interaction}}+\underbrace{2\nu\overline{\frac{\partial u_\alpha}{\partial x_\beta}\frac{\partial u_\gamma}{\partial x_\beta}}}_{\text{dissipation}}\underbrace{-\overline{\frac{p}{\rho}\left(\frac{\partial u_\alpha}{\partial x_\gamma}+\frac{\partial u_\gamma}{\partial x_\alpha}\right)}}_{\text{redistribution by pressure}}$$

$$\underbrace{+\frac{\partial}{\partial x_\beta}\left\{-\overline{u_\alpha u_\beta u_\gamma}+\overline{\frac{p}{\rho}(u_\alpha\delta_{\beta\gamma}+u_\gamma\delta_{\alpha\beta})}+\nu\frac{\partial R_{\alpha\gamma}}{\partial x_\beta}\right\}}_{\text{diffusion by velocity, pressure and viscosity}}$$



<u>Values of z_o in cm.</u>

mud flats, ice	10^{-3} to $3 \cdot 10^{-3}$
smooth sea	$2\ 10^{-2}$ to $3 \cdot 10^{-2}$
sand	10^{-2} to 10^{-1}
snow field	10^{-1} to $6 \cdot 10^{-1}$
mown grass (~1 cm)	10^{-1} to 1
low grass, steppe	1 to 4
fallow field	2 to 3
high grass	4 to 10
palmetto	10 to 30
suburbia	10^2 to $2 \cdot 10^2$
city	10^2 to $4 \cdot 10^2$

From NASA CR-2288 (1973)